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OPTIMUM PREVENTIVE MAINTENANCE POLICIES FOR THE AMRAAM MISSILE

THESIS

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THESIS

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Table of Contents

Page

List of Figures

Figure | Page

List of Tables

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Abstract

The overall objective of this research effort was to formulate a preventive maintenance strategy for AMRAAM missiles subject to extended captive carry flight time. A preventive maintenance policy is only applicable if the item in question is aging, or deteriorating with time. Therefore, a supporting objective of this research is to characterize the aging process of the missile system through a non-parametric analysis of its Mean Residual Life (MRL) function. Three non-parametric, censored-data MRL function estimation techniques discussed in the literature are examined via a numerical example. All three estimation techniques provide MRL functions that exhibit greatly exaggerated decreasing trends compared to the MRL function of the underlying distribution in the example. A semi-parametric technique for estimating the MRL function is developed that shows dramatic improvement over the non-parametric results. Although the MRL analysis of the current AMRAAM failure data failed to provide evidence that the missile system is aging, three preventive maintenance policies discussed in the literature are investigated. The traditional approach of preventive maintenance policy optimization via cost function minimization requires the cost of a system failure be explicitly known. However, the penalty for a system failure is often subjective and difficult to express in monetary terms. A "reliability cost model" is developed whereby system reliability for each policy is expressed as a function of cost. This technique allows a direct assessment of the trade-off between cost and projected system reliability. Theoretical results are presented and the performance of the model applied to empirical data is assessed via a Monte Carlo simulation.

OPTIMUM PREVENTIVE MAINTENANCE POLICIES FOR THE AMRAAM MISSILE

I. Introduction

Overview

The Advanced Medium Range Air-to-Air Missile (AMRAAM) is a highly capable active radar guided missile carried by USAF F-15 and F-16 fighters. The AMRAAM missile system is composed of four major components: 1) the seeker section, 2) the warhead / fuse section, 3) the flight control section, and 4) the rocket motor section. The missile has a built-in-test (BIT) capability whereby the major missile components are electronically monitored and failures are reported to the pilot. When a failure is indicated the missile is no longer launch capable.

Operational requirements over the last several years have caused the missiles to accumulate extensive "captive carry" hours whereby the missiles are carried on the fighters without being launched. The AMRAAM was initially designed for a captive carry mean time between failure (MTBF) of 1000 hours. However, a low frequency vibration problem discovered when the AMRAAM is captive carried on fuselage stations of the F-15 prompted the government to reduce the MTBF acceptance requirement to 450 hours. Due to current mission requirements most AMRAAMs being flown overseas have in excess of 450 hours and many have in excess of 1000 hours captive carry time and a large number of BIT indicated failures during captive carry operations have been recorded.

Williams and Pohl (1997) analyzed AMRAAM failure data and determined the life time distribution of the missile does not always exhibit a constant failure rate. In fact, several periods of increasing failure rate were found indicating the missiles age, or wear, with respect to cumulative captive carry time. However, the exact number of captive carry hours at which the missile's performance is degraded remains an open issue and no preventive maintenance policy is currently being used to improve performance.

Background

The objective of a preventive maintenance policy is to minimize system failures via replacement of components that reach a life at which the system's reliability gets to be lower than the reliability goal set for the next mission. (Kececioglu, 1995: xxviii) Reliability is defined as "the probability an item will adequately perform its specified purpose for a specified period of time under specified environmental conditions." (Leemis, 1995: 2) Two criteria must be met before a preventive maintenance policy is considered. First, the item in consideration must be aging, or deteriorating with time. If the item remains "as good as new" regardless of age (exponential lifetime), or is "used better than new" (improving with time), then preventive maintenance will not improve reliability. Second the cost of a system failure must be greater than the cost of performing preventive maintenance. If the penalty of an in operation failure is no greater than the penalty for performing scheduled maintenance then no cost benefit is realized even if reliability is improved. Given these criteria are met, the best preventive maintenance policy is one that optimizes some objective such as minimizing cost, maximizing availability, or maximizing reliability.

The reliability function, $S(t)$ (sometimes called the "survivor function") of an item is given by $S(t) = 1 - F(t)$ where $F(t)$ is the lifetime cumulative distribution function. When the item of interest is a system composed of several independent components such that the failure of any one component causes a system failure (a series system), the system reliability, $S_0(t)$ can be expressed in terms of the component reliabilities:

$$S_0(t) = \prod_{i=1}^{nc} S_i(t) \tag{1}$$

where $S_i(t)$ = reliability of component i

nc = number of components.

Typical maintenance models require specific knowledge of the survivor functions at both the system and component level. Estimates of the survivor functions must be made from available failure data that is often censored. Data censoring occurs when specific failure times are not known for all items on test. Right censoring occurs when an item is removed from test before failure for some reason. In this case only a lower bound is known for the item's lifetime. There are three types of right censoring. Type I censoring occurs when the test is terminated at some specified time, t_0. The failure times for all items that have not failed by this time are censored with a lower bound of t_0. Type II censoring occurs when the test is terminated at a specified number of failures. Type III, or random censoring, occurs when items are removed from test at any time. The current AMRAAM failure data set consists of a large number of observations ($n = 815$) subject to approximately 74% random right censoring. Future data is projected to have over 2500 observations subject to approximately 80% censoring.

Research Objectives

The overall objective of this research effort is to formulate a preventive maintenance strategy for using, retiring, or refurbishing AMRAAM missiles subject to extended captive-carry flight time. This objective requires assessment of the two aforementioned criteria for implementing a preventive maintenance policy. Namely, is the system aging and, if so, is the cost of an in-operation failure greater than the cost of performing preventive maintenance? We assume the answer to the second question is yes, even if the penalty for an in operation failure is not measured directly in dollar terms. The aging criterion is yet to be resolved. Williams and Pohl (1997) found evidence of aging through periods of increased hazard rate. However, the exponential distribution, which has a constant failure rate and is indicative of an item that does not age, was a good fit to the failure data. Therefore, a supporting objective of this research is to characterize the aging process of the missile system through a non-parametric analysis of its Mean Residual Life (MRL) function.

Scope

We make a simplifying assumption that there are infinite spares available. In other words, if a missile is taken out of service for maintenance, planned or otherwise, a spare missile is always available to fill mission requirements. Under this assumption preventive maintenance does not affect missile availability and the time required to perform preventive maintenance need not be considered in the cost models.

Overview of Subsequent Chapters

Chapter 2 contains a review of the existing literature covering the topics pertinent to this research. Non-parametric estimation of the MRL function and tests for aging are

explored. Several preventive maintenance policies with their associated cost models and optimization techniques are also examined. The methodology for the final analysis is developed in Chapter 3. Existing techniques for MRL analysis are validated and a technique for improving the non-parametric estimate of the MRL function with censored data over those described in the literature is developed. A nontraditional approach to maintenance policy optimization is developed whereby the objective is to maximize system reliability subject to an unspecified cost constraint. The methodologies developed in Chapter 3 are applied to the AMRAAM failure data set and results are reported in Chapter 4. Chapter 5 contains a summary of the thesis effort, including an overview and discussion of the impact of the results and ideas for future research.

II. Literature Review

Overview

This chapter provides an overview of the current literature in areas pertaining to this thesis. The chapter begins with a review of the MRL function and its applications. This review includes aging analysis, non-parametric estimation of the MRL function, and statistical tests for the decreasing mean residual life (DMRL) and new-better-than-used in expectation (NBUE) life distribution classes. The next section demonstrates the various MRL concepts via a numerical example that includes both complete and censored data cases. In the final section, three common preventive maintenance policies and their associated cost models are examined. This section concludes with a discussion of maintenance policy optimization.

Mean Residual Life

The MRL Function. The mean residual life (MRL) function, $m(t)$, is defined as the expected remaining lifetime of an item given the item has survived to time t. Let T be a random variable denoting the lifetime of an item with associated cumulative distribution function (cdf) $F(t)$, survivor function $S(t) = 1\text{-}F(t)$, density function $f(t)$, and hazard function $h(t) = f(t)/S(t)$. By definition, the MRL function can then be expressed as:

$$m(t) = E[T - t \mid T \geq t]. \tag{2}$$

Like $F(t)$, $f(t)$, and $h(t)$, the MRL function is unique to, and completely determines, the distribution of T. The MRL function can be expressed in terms of the other lifetime distribution representations and vice versa. A common representation of the MRL function is in terms of the survivor function as shown by equation (3).

$$m(t) = \frac{1}{S(t)} \int_t^{\infty} S(u)du \,. \tag{3}$$

Relationships between the MRL function and other lifetime distribution representations are shown in Table 1.

Table 1. Lifetime Distribution Representations (Leemis, 1995:55)

	$f(t)$	$S(t)$	$h(t)$	$m(t)$
$f(t)$	-	$\int_t^{\infty} f(u)du$	$\dfrac{f(t)}{\int_t^{\infty} f(u)du}$	$\dfrac{\int_t^{\infty} uf(u)du}{\int_t^{\infty} f(u)du} - t$
$S(t)$	$-S'(t)$	-	$\dfrac{-S'(t)}{S(t)}$	$\dfrac{1}{S(t)} \int_t^{\infty} S(u)du$
$h(t)$	$h(t)\exp\left(-\int_0^t h(u)du\right)$	$\exp\left(-\int_0^t h(u)du\right)$	-	$\dfrac{\int_t^{\infty} \exp\left(-\int_0^u h(y)dy\right)du}{\exp\left(-\int_0^t h(u)du\right)}$
$m(t)$	$\dfrac{1+m'(t)}{m(t)}\exp\left(-\int_0^t \dfrac{1+m'(u)}{m(u)}du\right)$	$\exp\left(-\int_0^t \dfrac{1+m'(u)}{m(u)}du\right)$	$\dfrac{1+m'(t)}{m(t)}$	-

Note that $m(0) = \int_0^{\infty} uf(u)du$ is the expected value of T.

The Weibull distribution is commonly used in lifetime modeling due to the flexibility afforded by its two parameters. The shape parameter β controls the shape of the distribution and allows for aging, memoryless (exponential), or improving properties. The scale parameter α controls the spread of the distribution across the time axis. Weibull distribution representations (Leemis, 1995:88) are shown in Table 2, and Weibull survival, density, hazard, and MRL functions with various shape parameters and scale parameter $\alpha = 1$ are plotted in Figure 1.

Table 2. Weibull Distribution Representations

$f(t)$	$\beta\alpha^{\beta}t^{\beta-1}e^{-(\alpha t)^{\beta}}$
$S(t)$	$e^{-(\alpha t)^{\beta}}$
$h(t)$	$\beta\alpha^{\beta}t^{\beta-1}$
$m(t)$	$\frac{e^{(\alpha t)^{\beta}}}{\alpha\beta}\Gamma\left(\frac{1}{\beta}\right)\left[1-I\left(\frac{1}{\beta},(\alpha t)^{\beta}\right)\right]$ where $I(y,x)=\frac{1}{\Gamma(y)}\int_0^x u^{y-1}e^{-u}du$
μ	$\frac{1}{\alpha\beta}\Gamma\left(\frac{1}{\beta}\right)$
σ^2	$\frac{1}{\alpha^2}\left\{\frac{2}{\beta}\Gamma\left(\frac{2}{\beta}\right)-\left[\frac{1}{\beta}\Gamma\left(\frac{1}{\beta}\right)\right]^2\right\}$

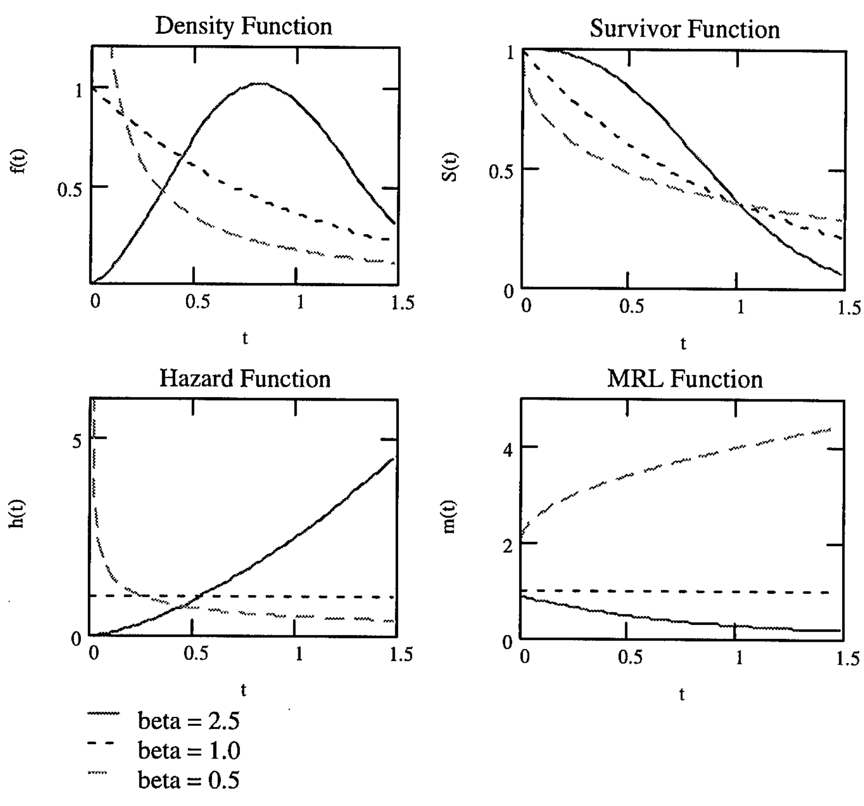


Figure 1. Weibull Lifetime Distribution Representations

A Weibull distribution with shape parameter $\beta = 1$ reduces to the exponential distribution. Note that in this case both the hazard function and the MRL function are constant. Shape parameter $\beta > 1$ indicates a deteriorating item. Note the hazard function is increasing and the MRL function is decreasing for the case $\beta = 2.5$. Finally, shape parameter $\beta < 1$ indicates an item that is improving with time. Note for the case $\beta = 0.5$ the hazard function is decreasing while the MRL function is increasing.

MRL and the Aging Process. Bryson and Siddiqui (1969: 1472) define aging as "the phenomenon whereby an older system has a shorter remaining lifetime, in some statistical sense, than a newer or younger one." The authors develop several criteria that provide statistical evidence of aging based on lifetime distribution classes. Among them are the increasing hazard rate (IHR) and DMRL functions. They define IHR as $h(t_2) \geq h(t_1)$ for all $t_2 \geq t_1 \geq 0$ and DMRL as $m(t_2) \leq m(t_1)$ for all $t_2 \geq t_1 \geq 0$. Recall plots of these functions for the Weibull distribution with shape parameter $\beta = 2.5$ are shown in Figure 1. "Net decreasing mean residual lifetime," more commonly called "new better than used in expectation" (NBUE), is another aging criterion specified by the authors. NBUE is related to the MRL function and defined as $m(0) \geq m(t)$ for all $t \geq 0$. Implications between these distribution classes are shown below.

$$\text{IHR} \Rightarrow \text{DMRL} \Rightarrow \text{NBUE}$$

Note the implications between the classes are unidirectional. The DMRL criterion is less restrictive than the IHR criterion in that the hazard rate may not be strictly increasing, while the MRL function associated with the same distribution is decreasing. For this reason Chen, Hollander, and Langberg (1993:120) conclude "the DMRL class may be more appropriate than the IHR class when the underlying physical process suggests wear-

out but, the failure rate is expected to fluctuate and not satisfy the IHR criterion." Similarly, the NBUE criterion is less restrictive than the DMRL criterion in that the MRL function may fluctuate and not be strictly decreasing, while the same distribution exhibits the NBUE characteristic.

One might further suppose that knowledge of the hazard rate at time t implies knowledge of the MRL at time t. Specifically, does $h(t) = 0$ imply $m(t) = 0$ and $h'(t) < 0$ imply $m'(t) > 0$? Muth (1975: 15) illustrates the somewhat surprising answer is no. He considers the distribution defined by the density, hazard, and MRL functions shown in Table 3. Plots of the hazard and MRL functions for this distribution are shown in Figure 2.

Table 3. Example Distribution Representations

$f(t)$	$\left[\left(1+2.3t^2\right)^2 - 4.6t\right]\exp\left(-t-\frac{2.3}{3}t^2\right)$
$h(t)$	$\frac{\left(1+2.3t^2\right)^2 - 4.6t}{1+2.3t^2}$
$m(t)$	$\frac{1}{1+2.3t^2}$

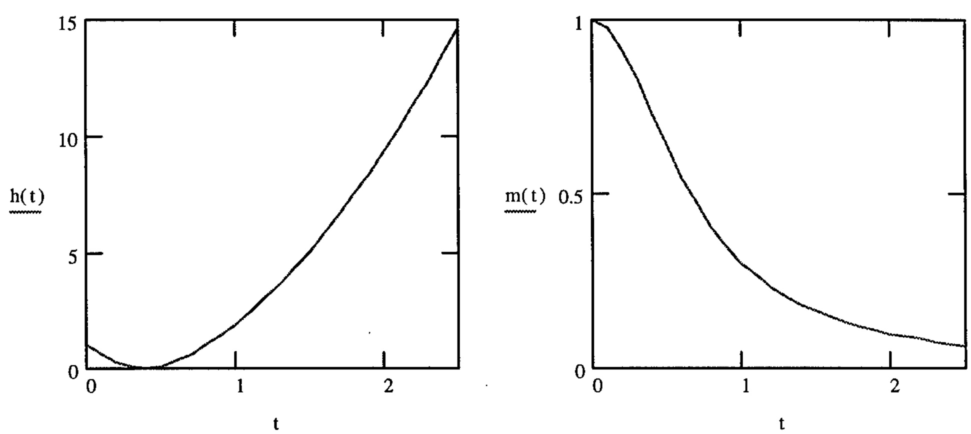


Figure 2. Hazard and MRL Functions for the Distribution Defined in Table 3

Note that hazard function for this distribution is not strictly increasing, yet it does have a decreasing MRL function. Furthermore, note that $h'(t) < 0$ in the interval [0, 0.35) while $m'(t) < 0$ in the same interval. Finally, note that $h'(0.35) = h(0.35) = 0$ while m(0.35) > 0 and $m'(0.35) < 0$. This example highlights an important difference between the hazard and MRL functions. Knowledge of the hazard function at time t provides information only about the immediate future, while knowledge of the MRL function at time t provides information about the complete future. Muth highlights this as another important advantage of the MRL function over the hazard function for characterizing the aging process of an item.

It is well known that the exponential distribution possesses the "memoryless" property. An item with an exponential lifetime distribution is always as good as new since, at any time t, its MRL is the same as at time zero. The item has no "memory" of the passage of time. Muth defines the concept of "virtual age" as a measure of memory. The virtual age of an item is "the amount by which the expected life has been reduced due to elapsed time." The virtual age, t_v, of an item can be expressed in terms of the MRL function:

$$t_v(t) = m(0) - m(t). \tag{4}$$

If the virtual age is zero, the item is defined has having "no memory." Note that for the exponential distribution t_v is zero for all t. As another example, consider an item with a Weibull lifetime distribution with shape parameter $\beta = 2.5$ and scale parameter $\alpha = 1.0$. The virtual age of the item at time $t = 1.0$ is

$$\begin{aligned} t_v(1) &= m(0) - m(1) \\ &= 0.887 - 0.289 = 0.598. \end{aligned}$$

This item does have memory, but it is not "perfect" in that $t_v(t) < t$. Muth denotes the special case of $t_v(t) = t$ as "perfect memory." Note that $0 \le t_v(t) \le \mu$ for all t for an item with a NBUE distribution and $t_v(t_2) \ge t_v(t_1)$ for all $t_2 \ge t_1$ if an item is characterized with a DMRL function.

Non-Parametric Estimation of the MRL Function. Suppose we desire to characterize the aging process of an item given a random sample from the item's unknown lifetime distribution $F(t)$. One approach to this problem is to fit a parametric distribution to the data and use the associated hazard and MRL functions to describe the aging process. The dangers inherent in this approach are well known, especially if the sample size is small (Leemis, 1995:252). Another approach is to directly characterize the aging process through non-parametric analysis of the MRL function. Bryson and Siddiqui (1969: 1485) demonstrate the value of plotting the empirical MRL function for this purpose. The sample MRL function has advantages over the sample density or hazard functions in that the sample MRL function smoothes fluctuations inherent in observed data (Muth, 1975:22).

The sample MRL function, $m_n(t)$, is formed by substituting the sample survivor function, $S_n(t)$, for $S(t)$ in (2) to obtain

$$m_n(t) = \frac{1}{S_n(t)} \int_t^{\infty} S_n(u)du . \qquad (5)$$

Guess and Proschan (1985:8-9) provide details for computing the sample MRL function. Their results for the case of no ties in the data are shown below.

$$m_n(t) = \frac{\sum_{i=k+1}^{n}(X_i - t)}{n-k} \quad \text{for} \quad t \in [X_k, X_{k+1}) \text{ and } k = 0, 1, \ldots, n\text{-}1 \qquad (6)$$

where n = number of sample observations;

X_i = i^{th} ordered observation where $X_{i-1} < X_i < X_{i+1}$;

$X_0 = 0$.

They also provide details for computing the empirical MRL when ties exist in the data not included here.

Chen, Hollander, and Langberg (1983) propose a sample MRL function for censored data by using the Kaplan-Meier estimate for the survival function, $S_{KME}(t)$. Let (Z_j, δ_j) be the ordered set of observations where $\delta_j = 0$ if observation j is censored and $\delta_j = 1$ if observation j is not censored. The Kaplan Meier estimate of the survival function is then

$$S_{KME}(t) = \begin{cases} 1 & , \ 0 \le t \le Z_1 \\ \prod_{j=1}^{k-1}\left(\frac{n-j}{n-j+1}\right)^{\delta_j} & , \ Z_{k-1} < t \le Z_k \\ 0 & , \ t > Z_n \end{cases} \qquad (7)$$

Substituting equation (7) for $S_n(t)$ in equation (5) the MRL function becomes

$$m_{KME}(t) = \frac{1}{S_{KME}(t)}\left\{\sum_{i=k}^{n-1} S_{KME}(Z_{i+1})(Z_{i+1} - Z_i) + S_{KME}(Z_k)(Z_k - t)\right\}, \ Z_{k-1} < t \le Z_k \qquad (8)$$

It is obvious from equations (6) and (8) that $m_n(t)$ and $m_{KME}(t)$ respectively are not smooth and have nodes at the sample observations. Kulasekera (1991) obtained a smooth MRL function from a smooth estimate of the distribution function, which in turn is derived from the kernel estimator of the density function. The kernel density estimator is constructed similar to a histogram whereby a "bump" called the kernel function, $K(x)$,

is centered over each observation. The kernel function is itself a density function that is typically chosen to be symmetric and mound shaped. The kernel estimate of the density function is formed by summing the kernel function "bumps."

In the case of complete data, Kulasekera expressed the sample distribution function in terms of the kernel density estimator to obtain

$$F_s(t) = \frac{1}{n}\sum_{j=1}^{n} W\left(\frac{t - X_j}{h}\right) \tag{9}$$

where $$W(t) = \int_{-\infty}^{t} K(u)du\,.$$

The parameter h used in equation (9) is called the "smoothing parameter" or "bandwidth" and determines the width of the kernel function bumps. As the value of h is increased the width of the kernel function bumps are increased thereby reducing the granularity of the density and corresponding distribution function estimates. Furthermore, Kulasekera observed equation (5) could be expressed:

$$m_n(t) = \frac{\overline{X} - \int_0^t (1 - F_n(u))du}{1 - F_n(t)} \quad , \qquad t > 0. \tag{10}$$

Finally, he derived the kernel estimate of the MRL function, $m_s(t)$, by substituting equation (9) in (10) to obtain

$$m_s(t) = \frac{\overline{X} - \int_0^t (1 - F_s(u))du}{1 - \frac{n}{n+1}F_s(t)}\,. \tag{11}$$

When the data are censored, Kulasekera uses the Kaplan Meier estimator (7) and the kernel density estimator to derive the estimated MRL function $m_{KMEs}(t)$. The derivation is analogous to the complete data case. The resulting estimate of survival function is

$$S_{KMEs}(t) = 1 - \sum_{j=1}^{n} (S_{KME}(Z_j) - S_{KME}(Z_{j+1}))W\left(\frac{t - Z_j}{h}\right). \tag{12}$$

Note that $S_{KME}(Z_j) - S_{KME}(Z_{j+1})$ is merely the size of the jump in the Kaplan Meier survival function at observation j. Using the result of equation (12) in equation (10), the smooth estimator of the censored data MRL is

$$m_{KMEs}(t) = \frac{\int_0^M S_{KME}(u)du - \int_0^t S_{KMEs}(u)du}{S_{KMEs}(t)}. \tag{13}$$

Kulasekera finds that the kernel estimators of the MRL are a much better approximation over the empirical MRL function for certain choices of the bandwidth, h.

Yet another technique for computing the MRL function with censored data is derived from the Piecewise Exponential Estimator (PEXE) of the survivor function introduced by Kim and Proschan (1991). The premise of the PEXE is to estimate the average failure rate in each interval between successive failures, fit an exponential survivor function to each interval, then piece the interval survivor function estimates together to obtain the system survivor function. The PEXE estimate of the hazard function is

$$h_n(t) = \frac{1}{\sum_{i=j}^{k-1} (n-i)(Z_{i+1} - Z_i)}, \quad Z_j < t \le Z_k \tag{14}$$

where Z_j = is the first observed failure $(\delta_j = 1) \le t$;

Z_k = is the first observed failure $(\delta_k = 1) > t$.

Note equation (14) is (observed number of failures) / (observed total time on test) for each interval between observed failures. It is well known that this is the form of the maximum likelihood estimate (MLE) of the failure rate for the exponential distribution. Let z_j, $j = 1\ldots m$ be the ordered set of observed failure times where $m \leq n$ is the total number of observed failures. The resulting PEXE survivor function is

$$S_{PEXE}(t) = \exp\left(-\sum_{k=1}^{i-1} h_n(z_k)(z_k - z_{k-1}) - h_n(z_i)(t - z_{i-1}) \right), \quad z_{i-1} < t \leq z_i. \qquad (15)$$

An important observation from equation (15) is that there is no attempt to extrapolate the estimate of the survivor function beyond the last observed failure time. The authors point out three key advantages of the PEXE over the KME of $S(t)$. First, $S_{PEXE}(t)$ is a continuous function, although it does have nodes at the observed failure times, whereas $S_{KME}(t)$ is a step function. Second, $S_{PEXE}(t)$ is responsive to the location of the censored observations between observed failures while $S_{KME}(t)$ is not. Third, the step function nature of $S_{KME}(t)$ tends to overestimate the underlying survivor function while $S_{PEXE}(t)$ does not. The PEXE estimate of the MRL function was proposed by Joe and Proschan (1981) and is formed by substituting $S_{PEXE}(t)$ of equation (15) in equation (3) to obtain

$$m_{PEXE}(t) = \frac{\int_t^{\infty} S_{PEXE}(u)\,du}{S_{PEXE}(t)} \qquad (16)$$

Complete Data Tests for MRL Distribution Classes. Suppose non-parametric analysis of observed data suggests the underlying distribution is DMRL or NBUE. There are several statistical tests described in the literature to formally test this observation. The boundary between DMRL or NBUE (deteriorating item) and an IMRL

or new worse than used in expectation (NWUE) (improving item) is the exponential life distribution which has a constant MRL function. Therefore we wish to test:

H_0: the life distribution is exponential

vs.

H_1: the life distribution has a DMRL and is not exponential

or

H_2: the life distribution is NBUE and is not exponential.

The dual classes of IMRL and NWUE are tested under the alternatives

H_1': the life distribution has an IMRL and is not exponential

and

H_2': the life distribution is NWUE and is not exponential

respectively.

Hollander and Proschan (1975) introduced the V^* test statistic to test the DMRL (IMRL) alternatives. They developed their test statistic by observing that the quantity

$$D(s,t) = S(s)S(t)[m(s)-m(t)] \tag{17}$$

is a weighted measure of the deviation of H_1 from H_0 for $s < t$. Note that $D(s,t) = 0$ if and only if H_0 is true. An average measure of the deviation of H_1 from H_0 is found by

$$\Delta(F) = \iint_{s<t} D(s,t)dF(s)dF(t). \tag{18}$$

Using the empirical estimates for the parameters in equation (18), they derived

$$V^* = \frac{\frac{1}{n^4}\sum_{i=1}^{n} c_{in} X_i}{\overline{X}} \tag{19}$$

where $$c_{in} = \frac{4}{3}i^3 - 4ni^2 + 3n^2 i - \frac{1}{2}n^3 + \frac{1}{2}n^2 - \frac{1}{2}i^2 + \frac{1}{6}i.$$

$n^{1/2}V^*$ is asymptotically $N(0, 1/210)$, thus the DMRL statistic is $(210n)^{1/2}V^*$. Large values of the test statistic indicate a DMRL while small values indicate an IMRL. Although not

presented here, the authors also develop a K^* statistic to test for distributions that are NBUE.

Aly (1990) proposed a test for NBUE based on the quantity

$$\gamma(S) = \int_0^\infty S(t)\left(1 + \ln S(t)\right)dt \,. \tag{20}$$

Note that $\gamma(S) = 0$ if H_0 is true and $\gamma(S) \geq 0$ under H_2. Equation (20) expressed in terms of F becomes

$$\gamma(F) = \int_0^\infty tJ\left(F(t)\right)dF(t) \tag{21}$$

where $$J(u) = 2 + ln(1\text{-}u).$$

The quantity $t_n = \gamma(F_n)$ is formed by substituting the empirical distribution F_n for F in equation (21). This quantity expressed in computational form is

$$t_n = \frac{1}{\overline{X}} \sum_{i=1}^{n} \left\{1 + \ln\left(1 - \frac{i-1}{n}\right)\right\}\left(1 - \frac{i-1}{n}\right)\left(X_i - X_{i-1}\right). \tag{22}$$

The statistic $n^{1/2}\, t_n / \overline{X}$ converges to a $N(0,1)$ random variable as the sample size n becomes large. Because the convergence is slow, a the author developed a modified form of the test statistic with a faster rate of convergence and shows it to be

$$T_n = \frac{n^{1/2}\left(t_n - \lambda_n \overline{X}\right)}{\sigma_n \overline{X}} \tag{23}$$

where $$\lambda_n = 1 + \frac{1}{n}\sum_{j=1}^{n} \ln\left(1 - \frac{j-1}{n}\right);$$

$$\sigma_n^2 = \frac{1}{n}\sum_{j=1}^{n} \left\{1 + \ln\left(1 - \frac{j-1}{n}\right)\right\}^2 .$$

H_o is rejected in favor of H_2 for large values of T_n and is rejected in favor of H_2' for small values of T_n. The author concludes the T_n statistic outperforms Hollander and Proschan's V^* statistic in terms of the Pitman asymptotic relative efficiency (ARE) for the Weibull distribution.

Ahmad (1992) proposes a new test for DMRL that he claims is easier to compute and performs better than Hollander and Proschan's V^* statistic. However, Kumazawa (1993) asserts this test is equivalent to Hollander and Proschan's K^* statistic for NBUE. The development of Ahmad's test is presented here since, as discussed later in this chapter, his work was extended to the censored data case.

Ahmad based his test on the intuitive notion that if $m(t)$ is decreasing and differentiable, then $m'(t) \leq 0$ for $t \geq 0$. The first derivative of the MRL function is given by

$$m'(t) = \frac{1}{S^2(t)}\left(f(t)\int_t^\infty S(u)du - S^2(t) \right) \tag{24}$$

Note that $m'(t) \leq 0$ if the quantity $\left(S^2 - f(t)\int_t^\infty S(u)du \right) \geq 0$. Therefore an average measure of the deviation of H_2 from H_0 is

$$\delta_F = \int_0^\infty \left(S^2(t) - f(t)\int_t^\infty S(u)du \right) dt. \tag{25}$$

Substituting the sample survivor function S_n for S, the symmetrized U-statistic form of equation (25) becomes

$$U_n = \frac{1}{n(n-1)} \sum_{i<j} (3X_i - X_j). \tag{26}$$

The quantity $n^{1/2}U_n$ is asymptotically $N(0, 1/3)$, thus the test statistic is $(3n)^{1/2}U_n$. H_o is rejected in favor of H_2 for large values of $(3n)^{1/2}U_n$ and is rejected in favor of H_2' for small values of $(3n)^{1/2}U_n$.

Censored Data Tests for MRL Distribution Classes. The three tests discussed thus far are limited in that they only apply when the failure data is complete. Fortunately, there are several tests for DMRL and NBUE that accommodate randomly censored data described in the literature. Chen, Hollander, and Langberg (1983) introduce such a test for DMRL. They extend the work of Hollander and Proschan by substituting the Kaplan-Meier estimator of the survival function (7) in equation (18) to obtain

$$\Delta(\hat{F}) = \sum_{i=1}^{n} \left\{ -\frac{1}{6}\prod_{j=1}^{i-1} c_j^{\delta_i} + \frac{1}{2}\prod_{j=1}^{i-1} c_j^{2\delta_i} - \frac{1}{3}\prod_{j=1}^{i-1} c_j^{4\delta_i} \right\}(Z_i - Z_{i-1}) \tag{27}$$

where $$c_j = \frac{n-j}{n-j+1}.$$

Analogous to the derivation of the V^* statistic, the V^c statistic for censored data is then

$$V^c = \frac{\Delta(\hat{F})}{\hat{\mu}} \tag{28}$$

where $$\hat{\mu} = \sum_{i=1}^{n} \left\{ \prod_{j=1}^{i-1} c_j^{\delta_j} \right\}(Z_i - Z_{i-1}).$$

The quantity $n^{1/2}V^c$ is asymptotically $N(0, \tau_0^2)$, thus the DMRL statistic is $n^{1/2}V^c/\tau_0$. A consistent estimator of τ_0^2 is shown to be

$$\hat{\tau}_0^2 = \frac{1}{720} + \sum_{i=1}^{n-1} \frac{n}{(n-i+1)(n-i)} \left\{ \frac{B_i(2)}{72} - \frac{B_i(3)}{18} + \frac{B_i(4)}{16} + \frac{B_i(5)}{45} - \frac{B_i(6)}{18} + \frac{B_i(8)}{72} \right\}$$
$$+ n\left\{ \frac{B_n(2)}{72} - \frac{B_n(3)}{18} + \frac{B_n(4)}{16} + \frac{B_n(5)}{45} - \frac{B_n(6)}{18} + \frac{B_n(8)}{72} \right\} \tag{29}$$

where $$B_i(a) = \exp\left(\frac{-a}{\hat{\mu}} Z_i\right), \quad i = 1, \ldots, n.$$

When ties exist in the data between censored and uncensored observations, treat the uncensored values as preceding the censored values when forming the ordered list of Z_i values. The authors caution that no more than 50% of the observations should be censored when applying the V^c test. However, they exercise the test in an example where the data is 57% censored with good results.

Lim and Koh (1996) extend the work of Aly to accommodate randomly censored data by using the Kaplan-Meier estimate $F_{KME} = 1 - S_{KME}$ of F in equation (21) to obtain the test statistic

$$L_n^c = \frac{\gamma(F_{KME})}{\hat{\mu}} \tag{30}$$

where $$\gamma(F_{KME}) = \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} c_j^{\delta_j} \right) \left(\ln\left(\prod_{j=1}^{i-1} c_j^{\delta_j} + 1 \right) \right) (Z_i - Z_{i-1});$$

c_j and $\hat{\mu}$ defined as in equations (27) and (28) respectively.

The quantity $n^{1/2} L_n^c$ is asymptotically $N(0, \sigma^2)$, thus the DMRL test statistic is $n^{1/2} L_n^c / \sigma$. The authors show a consistent estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{4} + \left\{ \sum_{i=1}^{n-1} \frac{n}{(n-i+1)(n-i)} \left[\frac{1}{4} - \frac{Z_i}{2\hat{\mu}} + \frac{Z_i^2}{2\hat{\mu}^2} \right] B_i(4) \right\}$$
$$- n \left[\frac{1}{4} - \frac{Z_n}{2\hat{\mu}} + \frac{Z_n^2}{2\hat{\mu}^2} \right] B_n(4) \tag{31}$$

where $B_i(a)$ is defined as in equation (29). The authors find the L_n^c statistic compares favorably with Chen Hollander and Lang's V^c statistic only when the data is "slightly" censored.

Lim and Park (1993) offer a test for DMRL with randomly censored data based on the work of Ahmad. They extend Ahmad's work by substituting the Kaplan-Meier estimate $F_{KME} = 1 - S_{KME}$ of F in equation (25). The computational form of the resulting test statistic is

$$\delta_n^c = \frac{1}{\hat{\mu}}\left\{\sum_{i=1}^{n}\left[2\prod_{j=1}^{i-1}(c_j)^{2\delta_j} - \prod_{j=1}^{i-1}(c_j)^{\delta_j}\right](Z_i - Z_{i-1})\right\} \tag{32}$$

where c_j is as defined in equation (27). The quantity $n^{1/2}\delta_n^c$ is asymptotically $N(0, \tau_0^2)$, thus the DMRL test statistic is $n^{1/2}\delta_n^c/\tau_0$. A consistent estimator of τ_0^2 is shown to be

$$\hat{\tau}_0^2 = \frac{1}{6} + \sum_{i=1}^{n-1}\frac{n}{(n-i+1)(n-i)}\left\{B_i(4) - \frac{4}{3}B_i(3) + \frac{1}{2}B_i(2)\right\}$$
$$- n\left\{B_n(4) - \frac{4}{3}B_n(3) + \frac{1}{2}B_n(2)\right\} \tag{33}$$

where $B_i(a)$ defined as in equation (29).

H_o is rejected for significantly large (small) values of $n^{1/2}\delta_n^c/\tau_0$ in favor of H_1 (H_1'). At issue is whether this test is more appropriate for DMRL or NBUE. Although the authors claim this is a test for DMRL, Ahmad's test from which it was derived was later proven to be a test for NBUE. Fortunately, the matter is inconsequential for this research effort as both DMRL and the less restrictive NBUE class satisfy the aging requirement.

MRL Numerical Example

The purpose of this section is to demonstrate the MRL concepts described above via a numerical example. I used the Weibull distribution with shape parameter $\beta = 1.5$ and scale parameter $\alpha = 1.0$ to generate $n = 100$ random variates used as the "observed" data set for the example. This technique allows easy comparison of the non-parametric

analysis to the true underlying distribution. MathSoft's Mathcad 6.0 software (Appendix A) was used to generate the random variates, perform all calculations, and generate the plots illustrated in this section. The ordered complete data set is shown in Table 4. The example is considered in two parts. The complete data case is examined first followed by results for randomly right-censored data.

Table 4. Weibull (1, 1.5) Ordered Data Set

Obs. #	X_i	Obs. #	X_i	Obs. #	X_i	Obs. #	X_i
1	0.0218	26	0.4442	51	0.7368	76	1.3439
2	0.0511	27	0.4503	52	0.7536	77	1.3848
3	0.0666	28	0.456	53	0.7576	78	1.3926
4	0.1049	29	0.46	54	0.7578	79	1.3984
5	0.1267	30	0.4735	55	0.7818	80	1.4511
6	0.1393	31	0.4888	56	0.7955	81	1.4676
7	0.2598	32	0.5033	57	0.8061	82	1.4766
8	0.2801	33	0.508	58	0.8265	83	1.5128
9	0.3022	34	0.5125	59	0.8416	84	1.5209
10	0.3078	35	0.5268	60	0.848	85	1.5269
11	0.3091	36	0.5336	61	0.851	86	1.5414
12	0.3119	37	0.5601	62	0.8594	87	1.5422
13	0.3129	38	0.6231	63	0.8821	88	1.5634
14	0.3155	39	0.6366	64	0.8987	89	1.5684
15	0.3197	40	0.6387	65	0.8993	90	1.6375
16	0.3197	41	0.6404	66	0.901	91	1.6544
17	0.3363	42	0.6559	67	0.9856	92	1.6779
18	0.3379	43	0.66	68	1.0324	93	1.6955
19	0.3441	44	0.6775	69	1.1028	94	1.7888
20	0.3593	45	0.6777	70	1.1234	95	2.0166
21	0.3907	46	0.7051	71	1.1613	96	2.3462
22	0.3957	47	0.7103	72	1.1755	97	2.8135
23	0.4328	48	0.7191	73	1.1837	98	2.8182
24	0.4337	49	0.7251	74	1.2054	99	3.4563
25	0.4433	50	0.7363	75	1.2368	100	3.5434

Complete Data.

Empirical MRL Functions. Equation (6) was used to generate the empirical MRL function and a smooth approximation of the MRL function was obtained using Kulasekera's result (11). The Epanechnikov kernel function (Silverman, 1986: 43)

$$K(t) = \frac{3}{4\sqrt{5}}\left(1 - \frac{1}{5}t^2\right), \qquad |t| < \sqrt{5} \tag{34}$$

$$0, \qquad \text{otherwise}$$

was used for the smooth approximation. Figure 3 shows a plot of the normal and smoothed empirical MRL functions contrasted with the true MRL function of the underlying distribution. A smoothing parameter $h = 0.1$ was used in this case. Figure 4 shows the same plots, but with smoothing parameter $h = 0.2$ to demonstrate its effect on the resulting MRL function approximation.

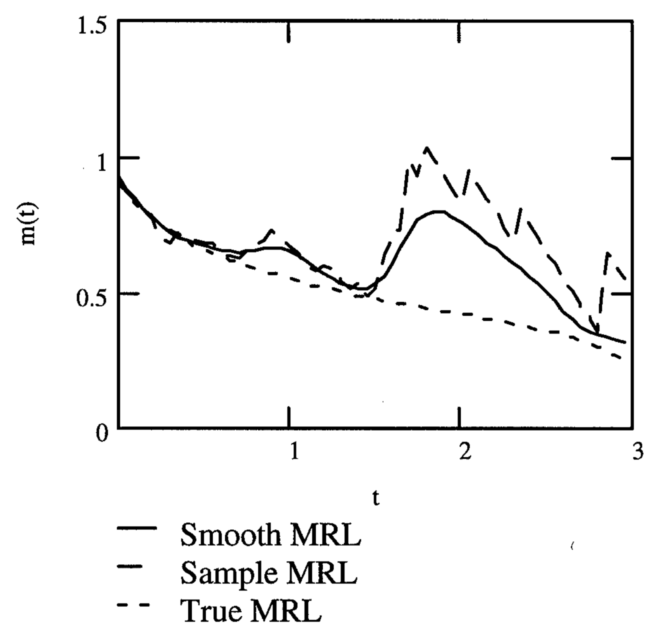


Figure 3. Normal and Smoothed ($h = 0.1$) MRL Functions

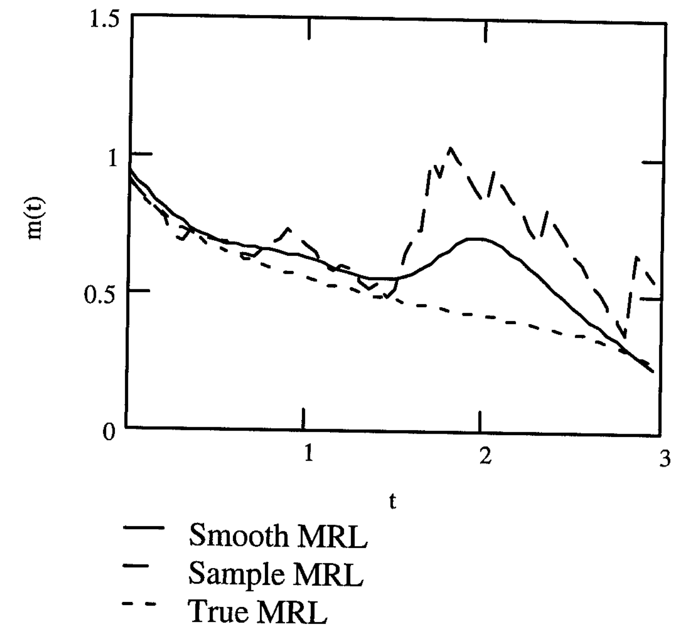


Figure 4. Normal and Smoothed ($h = 0.2$) MRL Functions

The overall impression of the empirical MRL functions is that the underlying distribution does not have a strictly decreasing MRL function, but may posses the NBUE characteristic. Note the empirical MRL functions are a fair approximation of the true MRL function for $t < 1.5$. When $t > 1.5$, the estimated MRL functions over estimate the true MRL function in all cases. Examination of the sample survivor curve shown in Figure 5 offers some insight into this phenomenon. Note that the sample survivor function underestimates the true survivor function in the vicinity $1.5 \leq t \leq 2$ while the tail of the survivor function is exaggerated for $t > 2$. Therefore, the numerator of equation (5) is over estimated while the denominator is under estimated causing the poor performance of the empirical MRL functions in this area. Also note that progressive smoothing of the empirical MRL function results in increased loss of detail, but can lead to a better approximation if the fluctuations are suspect not to be inherent in the true MRL function.

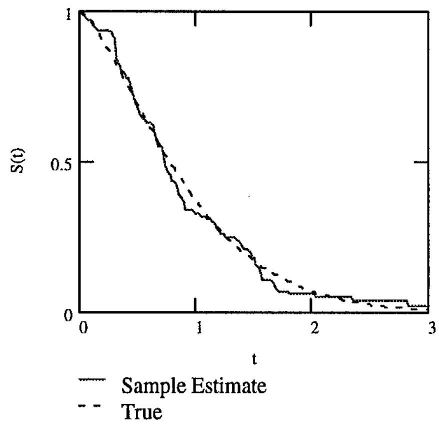


Figure 5. Complete Data Survivor Function

Statistical Tests. Hollander and Proschan's V^* test (16) was applied to the data set to test for DMRL. We would expect the test to favor of H_1 over H_0 since we know the underlying parametric distribution is DMRL. The resulting test statistic value $(210n)^{1/2}V^* = 1.289$ yields a p-value = 0.099. This test does not strongly confirm the DMRL of the true underlying distribution, but does agree with the impression of the empirical MRL estimate.

Aly's T_n test (23) and Ahmad's U_n test (26) were also applied to test for NBUE. The resulting test statistics were 2.926 and 4.008, with corresponding p-values of 0.0017 and 3.07×10^{-6} respectively. These tests strongly confirm the NBUE characteristic of the underlying distribution and the impression of the empirical MRL functions.

Censored Data. The data in Table 2 was randomly censored using the method described by Kulasekera (1990:100). Let $Y_1,\ldots,Y_n$ be a random sample from the censoring distribution G. Then the randomly right censored data is given by the pairs (Z_i, δ_i) where

$$Z_i = \min(X_i, Y_i)$$

$$\delta_i = 1 \text{ if } X_i \le Y_i \text{ and } 0 \text{ if } X_i > Y_i.$$

I used an exponential censoring distribution with rate parameter $\lambda = 0.8$ to obtain 43% censoring of the data set. Table 5 shows the resulting ordered (Z_i, δ_i) random censored data pairs.

Table 5. Censored Data Ordered Pairs

Obs. #	Z_i	δ_i	Obs. #	Z_i	δ_i	Obs. #	Z_i	δ_i	Obs. #	Z_i	δ_i
1	0.0218	1	26	0.3119	1	51	0.5268	1	76	0.7578	1
2	0.0511	1	27	0.3129	1	52	0.5601	1	77	0.7818	1
3	0.0566	0	28	0.3155	1	53	0.5802	0	78	0.7955	1
4	0.0619	0	29	0.3197	1	54	0.5814	0	79	0.8018	0
5	0.0666	1	30	0.3197	1	55	0.5968	0	80	0.8265	1
6	0.1043	0	31	0.3363	1	56	0.6186	0	81	0.8435	0
7	0.1049	1	32	0.3379	1	57	0.6255	0	82	0.848	1
8	0.1158	0	33	0.3441	1	58	0.6287	0	83	0.851	1
9	0.1267	1	34	0.3593	1	59	0.6387	1	84	0.8561	0
10	0.1383	0	35	0.37	0	60	0.6404	1	85	0.8821	1
11	0.1392	0	36	0.3798	0	61	0.6431	0	86	0.8852	0
12	0.1693	0	37	0.3907	1	62	0.6559	1	87	0.8987	1
13	0.1823	0	38	0.3957	1	63	0.6775	1	88	0.9776	0
14	0.1838	0	39	0.3974	0	64	0.6777	1	89	0.9937	0
15	0.1961	0	40	0.4318	0	65	0.6855	0	90	1.0357	0
16	0.2014	0	41	0.4433	1	66	0.6953	0	91	1.1378	0
17	0.2379	0	42	0.4442	1	67	0.7051	1	92	1.1613	1
18	0.2598	1	43	0.4503	1	68	0.7103	1	93	1.1837	1
19	0.2665	0	44	0.456	1	69	0.712	0	94	1.4511	1
20	0.2719	0	45	0.46	1	70	0.7191	1	95	1.4676	1
21	0.2752	0	46	0.4735	1	71	0.7251	1	96	1.5269	1
22	0.2801	1	47	0.4888	1	72	0.7363	1	97	1.5422	1
23	0.2818	0	48	0.5047	0	73	0.7368	1	98	1.5634	1
24	0.3027	0	49	0.508	1	74	0.7409	0	99	1.7888	1
25	0.3078	1	50	0.5125	1	75	0.7518	0	100	1.8518	0

Empirical MRL Functions. The KME (8), smoothed KME (13), and the PEXE (15) of the MRL function were applied to the censored data set. Plots of these MRL function approximations contrasted with the complete data MRL function and the true MRL function are shown in Figure 6. The complete data MRL function is included so an assessment can be made regarding the affect of the censoring. The censored data approximations can not be reasonably expected to perform better than the complete data

approximation. A smoothing parameter $h = 0.1$ was chosen arbitrarily for the smoothed KME function and the same kernel (24) was used as in the complete data case. Note that of each of the censored data empirical MRL functions depart significantly from the known parametric result in that the DMRL is greatly exaggerated. The departure from the complete data MRL function is even more dramatic when $t > 1.5$.

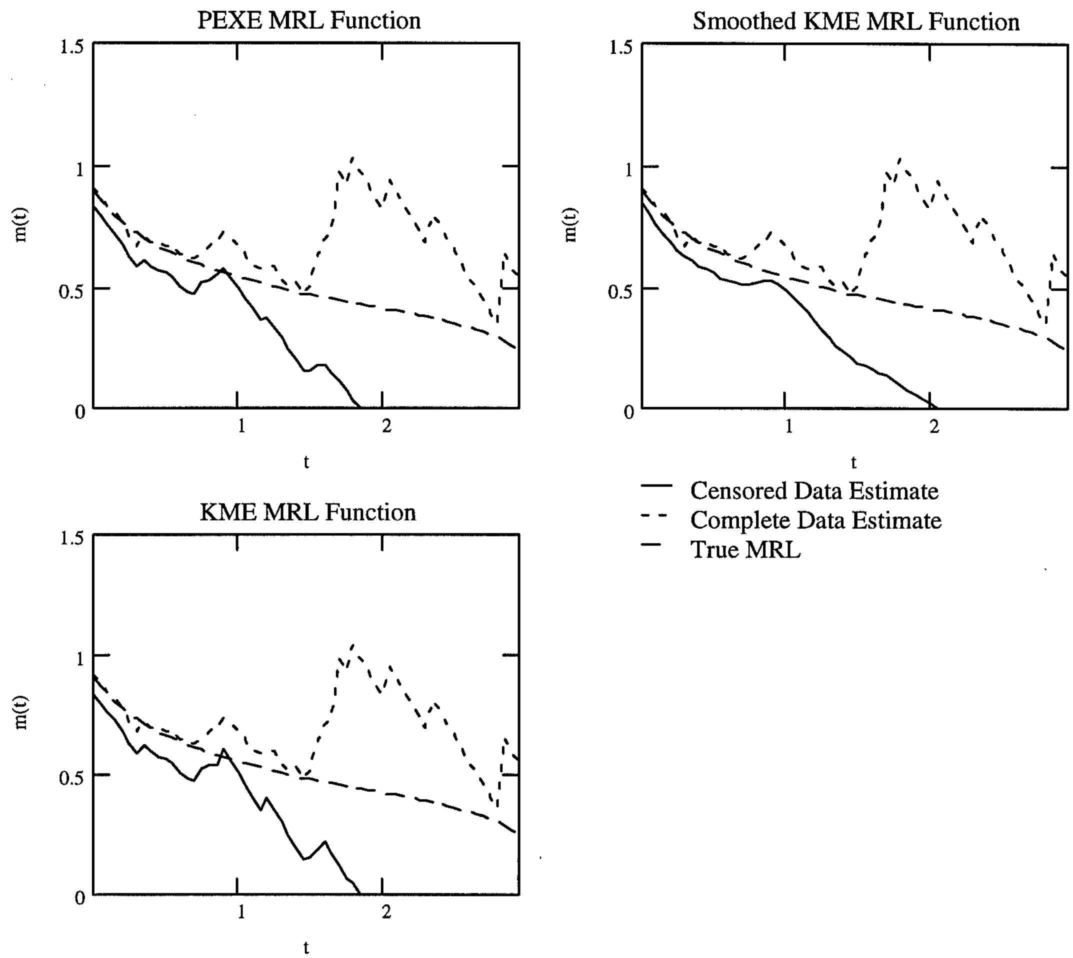


Figure 6. Censored Data MRL Estimates

Recall the empirical estimates of the MRL function are an application of equation (3) with empirical estimates for $S(t)$. Therefore, examination of the empirical survival functions should offer some insight to the decreased performance of the censored data empirical MRL estimates. Figure 7 shows the PEXE, KME, and smoothed KME

survivor functions. Note that each of these functions fail to preserve the "tail" of the true survivor function. In contrast to the complete data set where at least sparse information existed in the tail of the distribution, the censored data set contains no information for $t > 1.9$. This highlights a major limitation of each of the empirical survivor function and corresponding MRL function estimates described in the literature. None of the estimates attempt to extrapolate or estimate the survivor function past the last observation. Since $m(t) = 0$ when $S(t) = 0$, the truncated survivor functions cause the empirical MRL functions to drop off faster resulting in a more pronounced DMRL. This effect is exaggerated as either the amount of censoring increases or the sample size decreases. A "semi-parametric" technique to solve this problem is developed in Chapter 3.

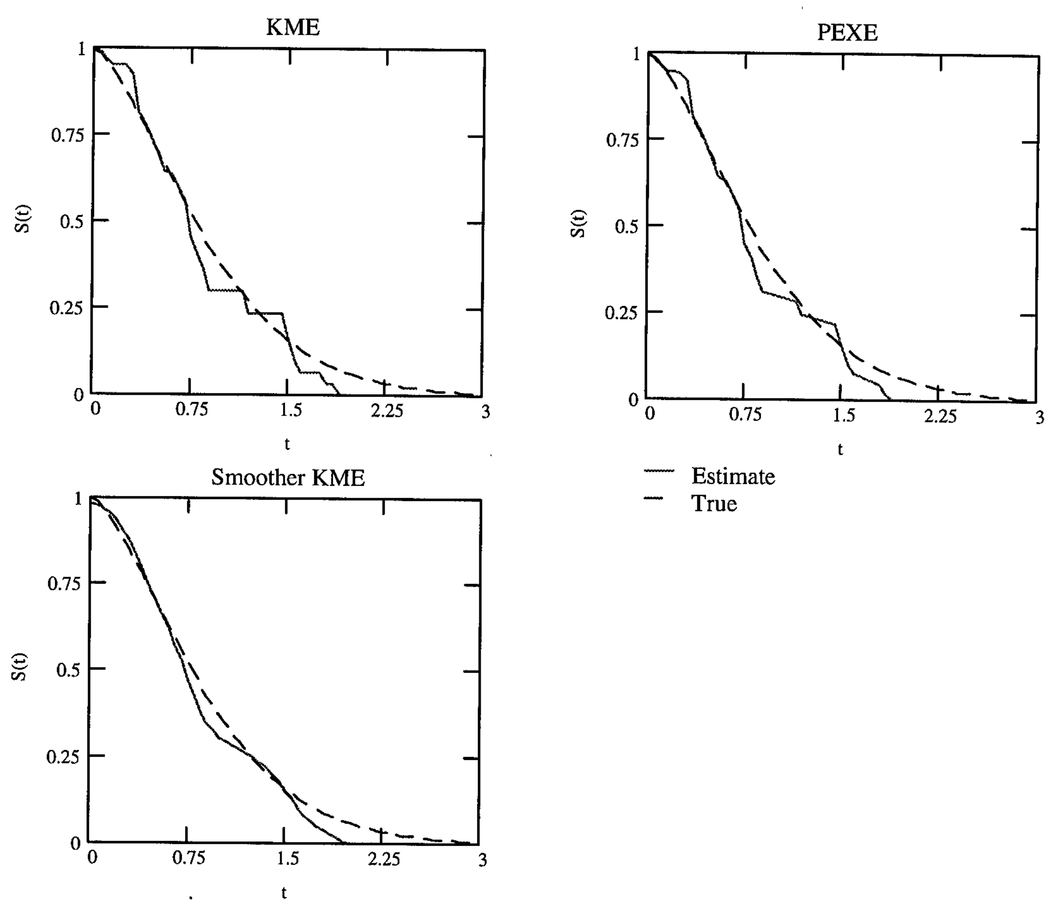


Figure 7. Censored Data Survivor Function Estimates

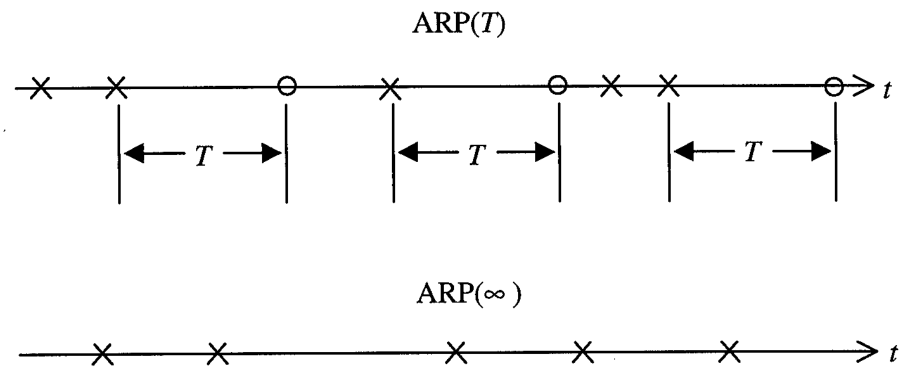
Statistical Tests. Chen Hollander and Langberg's test was applied to the censored data to check for DMRL. The resulting test statistic value $n^{1/2}V^c/\tau_0 = 1.086$ with a corresponding p-value = 0.139. This test did not confirm the DMRL of the underlying distribution even though the empirical MRL functions suggest a strongly decreasing MRL. Lim and Koh's test for NBUE resulted in a test statistic $n^{1/2}L_n^c/\sigma = 8.407$ with a corresponding p-value << 0.001. Lim and Park's test for NBUE resulted in a test statistic $n^{1/2}\delta_n^c/\tau_0 = 4.745$ with a corresponding p-value < 0.001. These tests strongly confirm the NBUE characteristic of the underlying distribution and the impression of the empirical MRL functions.

Preventive Maintenance

Policy Descriptions. Three preventive maintenance policies that have been extensively covered in the literature are the Age Replacement Policy (ARP), the Block Replacement Policy (ARP), and the Opportunistic Replacement Policy (ORP). The ARP is common when considering simple systems whereas the BRP and the ORP are appropriate for complex systems. The term "replaced" is synonymous with the expression "repaired to as good as new" for the following discussion. Barlow and Hunter (1960) define the ARP (Policy I) whereby the system is replaced upon failure or after a time T since the last failure, whichever occurs first. In the limit as T approaches infinity, the ARP reduces to replacing the system upon failure and no preventive maintenance is performed. Figure 8 illustrates representative ARP time sequences.

Barlow and Hunter also define a BRP (Policy II) whereby the system is replaced when the time t is a multiple of the replacement period T, that is when $t = kT$ where $k =$ 1,2,3,..., regardless of the number of intervening failures. Upon failure a "minimal

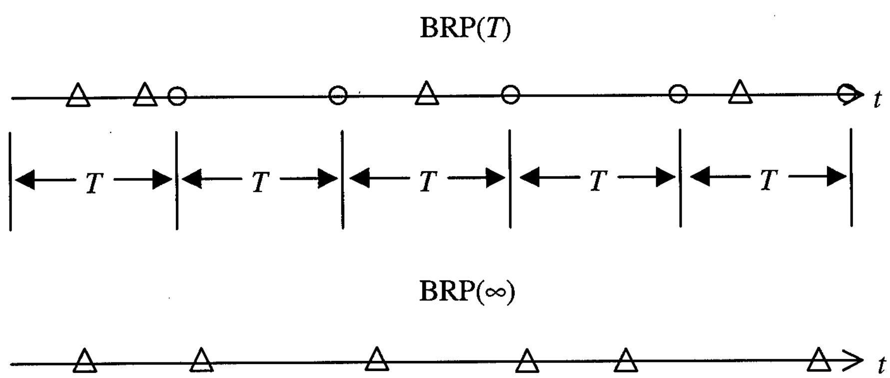
repair" is made whereby only the failed component is replaced. After such a minimal repair the authors assume the system failure rate is not significantly disturbed due to the aging of the other components. In the limit as the replacement period T approaches infinity, the BRP reduces to a policy of replacing individual components as they fail. No preventive maintenance is performed and the system is never returned to "as good as new" condition. Figure 9 illustrates representative BRP time sequences.



O Scheduled System Preventive Maintenance

X Failure with complete Repair

Figure 8. Age Replacement Policy



O Scheduled System Preventive Maintenance

Δ Failure with minimal Repair

Figure 9. Block Replacement Policy

The ORP was introduced by Radner and Jorgenson (1963) and modified by Gertsbakh (1977). The ORP is a combination of the ARP and BRP policies and may be beneficial when the cost of replacing the system is less than sum cost of replacing individual components. Under the ORP individual failed components are replaced if a failure occurs within time τ, the system is replaced if a failure occurs within the interval τ to T, and the system is replaced at time T since the last system replacement if no failures have occurred. Note that the ORP reduces to the ARP if $\tau = 0$, and equates to the BRP if $\tau = T$. Figure 10 illustrates representative ORP time sequences.

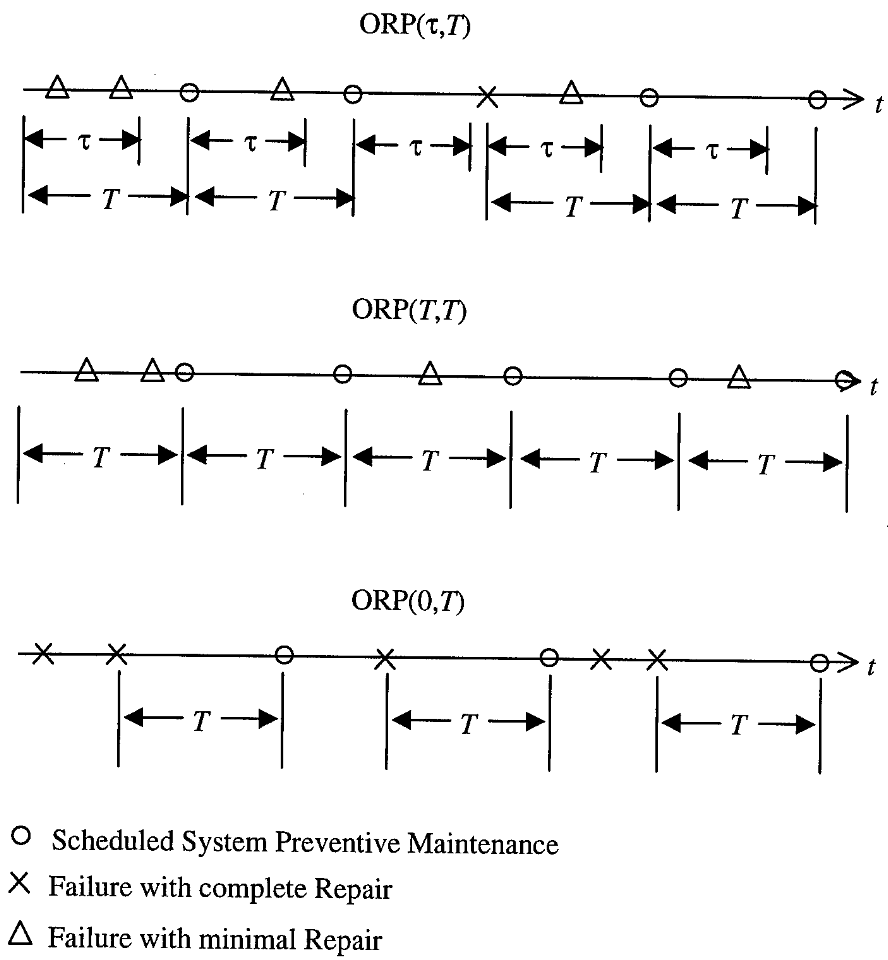


Figure 10. Opportunistic Replacement Policy

Of the three policies, the BRP is generally considered the easiest to implement since the preventive maintenance does not depend on the system failure history and can be scheduled in advance. Another advantage of this policy is that no documentation of failures or time in service of the system or any of the components is required for its implementation. The ARP is slightly more difficult to implement than the BRP in that system time in service must be tracked to determine when preventive maintenance must be performed in the event of no failures. The ORP is the most difficult of all to implement since system time in service and the parameters τ and T must be tracked to determine the appropriate maintenance action to take. These considerations are not directly considered in the cost models, but may be a criteria for deciding on a policy when two or more have otherwise equivalent costs.

Policy Costs. The cost of implementing each policy must be determined in order for the best policy to be chosen for a given situation. Let c_f be the cost of a system failure and c_p be the cost of a planned system replacement where $c_f > c_p$. The long run average cost per unit time of the ARP in terms of the replacement period T, $C_A(T)$ is well documented in the literature and is given by

$$C_A(T) = \frac{c_f \overline{S}_0(T) + c_p S_0(T)}{\int_0^T S_0(t)dt} \tag{35}$$

where $S_0(t)$ is the system survivor function.

The denominator of equation (35) is the expected time of each replacement period. This equation is easily solved given $S_0(t)$. When the system lifetime distribution is not explicitly known, a non-parametric cost model for the ARP can be obtained with an

estimate of $S_0(t)$ from failure data. In the limit as T approaches infinity the average cost per unit time of the ARP becomes

$$C_A(\infty) = \frac{c_f}{\mu}. \tag{36}$$

To determine the cost of the BRP and ORP policies, the cost of replacing individual components must be considered. Spearman (1986) defines the costs c_f and c_p as follows

$$c_f = c_b + c_s + \sum_{i=1}^{nc} c_i \tag{37}$$

$$c_p = c_s + \sum_{i=1}^{nc} c_i \tag{38}$$

where $c_b =$ cost (penalty) of a system breakdown

$c_s =$ setup repair cost

$c_i =$ cost to replace component i

$nc =$ number of components in the system.

Note that under this formulation the assumption $c_f > c_p$ holds for $c_b > 0$. Also, the cost of system failure followed by a minimal repair of the failed component only is given by

$$c_{fi} = c_b + c_s + c_i. \tag{39}$$

It should also be noted that the total cost of replacing individual components is greater than the cost of simultaneously replacing all components only if $c_s > 0$.

Spearman shows the long run average cost per unit time of the BRP in terms of the replacement period T, $C_B(T)$ is

$$C_B(T) = \frac{c_p + \sum_{i=1}^{4} (c_{fi}) W_i(T)}{T} \tag{40}$$

where $W_i(t) = E[N_i(t)]$ is the expected number of renewals of component i by time t.

The renewal function $W_i(t)$ must be determined for each component of the system to solve equation (40). $W_i(t)$ is found via the fundamental renewal equation (Barlow and Proschan, 1965: 50) shown in equation (41).

$$W_i(t) = \int_0^t \left(1 + W_i(t-x)\right) dF_i(x) . \tag{41}$$

Equation (41) is difficult to solve for many distributions, including the Weibull. Spearman develops an easy to implement approximation of the renewal equation for lifetime distributions with unbounded increasing failure rates (including the Weibull distribution) given by

$$W(t) \approx \max\{W_1(t), F(t)\} \tag{42}$$

where $$W_1(t) = \left(1 + \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2}\right) F(t) - \frac{1}{\mu}\int_0^t sf(s)ds .$$

When the component lifetime distributions $F_i(t)$ and $f_i(t)$ are not explicitly known, they must be estimated from failure data to apply equation (42). Either parametric or non-parametric estimates may be used. In the limit as T approaches infinity the average cost per unit time of the BRP becomes

$$C_B(\infty) = \sum_{i=1}^{n} \frac{c_{fi}}{\mu_i} . \tag{43}$$

The long run average cost per unit time of the ORP, $C_O(\tau,T)$ is similar to the BRP, but the denominator of equation (40) must be modified since the average duration of each replacement period is no longer fixed at T. Spearman develops the expected cost of the ORP and shows it to be

$$C_O(\tau,T)=\frac{c_p+\sum_{i=1}^{4}(c_{fi})W_i(\tau)}{\tau+\int_0^{T-\tau}\left[1-F_V(\tau,u)\right]du} \tag{44}$$

where $F_V(t_0,t)$ is the distribution function for the random variable V, the time until the next system renewal given the system is at time t_0.

The numerator of equation (44) is similar to that for the BRP (40). The approach to finding a solution to the fundamental renewal equation via the approximation (42) is unchanged. To solve the denominator, Spearman assumes Weibull component lifetime distributions and develops an estimate for the quantity 1- $F_V(\tau,u)$:

$$1-F_V(\tau,u)\geq\exp\left(\sum_{i=1}^{4}-\left(\frac{\tau+u}{\alpha_i}\right)^{\beta_i}+\left(\frac{\tau}{\alpha_i}\right)^{\beta_i}\right) \tag{45}$$

where α_i = Weibull scale parameter for component i;

β_i = Weibull shape parameter for component i.

This expression must be numerically integrated to calculate the denominator of the ORP cost equation.

Maintenance Policy Optimization. The problem of optimizing the aforementioned maintenance policies has received extensive attention in the literature. Almost without exception, the literature defines "optimal" in terms of minimizing cost. The cost equations for the respective policies are minimized with respect to the replacement period T (τ and T for the ORP) to find the value T^* providing the lowest cost for each policy. Once each policy is optimized, they can be directly compared to determine the overall best policy for a given situation. This approach to finding an optimal preventive maintenance policy requires the cost parameters described in equations (37) and (38) to

be well defined. This is not always the case. Consider a system such as the AMRAAM where the penalty of system failure, c_b, is degraded mission accomplishment or, in the extreme case, loss of life. In this case the cost c_b, while obviously high warranting consideration of a preventive maintenance policy, is subjective and its relative weight compared to the other cost parameters (expressed in dollars) is difficult to assess. The traditional approach of optimization through minimizing the cost function therefore can not be directly applied. Instead, we will attempt to express system reliability for each policy as a function of cost. A direct comparison can then be made between the policies to determine which policy provides the greatest reliability for a given cost (or conversely, given a specified reliability the "optimal" policy is the one with the lowest associated cost). The methodology for this approach will be developed in chapter 3.

III. Methodology

Overview

In this chapter the methodology to analyze the AMRAAM data set is defined. The chapter begins with a look at the MRL function. A technique to improve the empirical MRL functions discussed in chapter 2 is developed and demonstrated. The behaviors of the various DMRL / NBUE statistical tests are also examined with respect to various Weibull shape parameters, sample sizes, and censoring levels. In the next section, the preventive maintenance reliability cost model is developed and demonstrated. A cost parameter sensitivity analysis is performed with the model to determine the best policy with respect to parameter ratios. Finally the performance of the empirical reliability cost model is assessed with a numerical example.

Mean Residual Life

Semi-parametric MRL Function. The empirical MRL function estimates discussed in Chapter 2 are an application of equation (3) with empirical estimates for $S(t)$. However, the empirical survivor functions generally fail to preserve the "tail" of the distribution as the amount of censoring increases. The truncated survivor functions result in empirical MRL function estimates with an exaggerated DMRL. This effect is unacceptable for two reasons. First, we are attempting to characterize the aging process to determine if a preventive maintenance policy is applicable. An exaggerated MRL function may falsely indicate that a preventive maintenance policy is warranted when, in fact, the MRL of the underlying distribution may not support this conclusion. Second, an

accurate MRL function is an integral piece of the reliability cost model developed later in this chapter.

A method is required to extrapolate the empirical survivor functions, $S_n(t)$, past the last observation in order to improve the empirical MRL estimates. We propose a "semi-parametric" technique whereby a distribution is fitted to the data and the corresponding parametric survivor function, $\hat{S}(t)$, is used to estimate the survivor function beyond the last observation. The general form of semi-parametric survivor function, $\hat{S}_n(t)$ is

$$\begin{aligned} \hat{S}_n(t) &= S_n(t)\ , \qquad t \le Z_n \\ &= \hat{S}(t)\ , \qquad t > Z_n. \end{aligned} \tag{46}$$

The general form of the semi-parametric MRL function $\hat{m}_n(t)$ is then

$$\hat{m}_n(t) = \frac{1}{S_n(t)}\left(\int_t^{Z_n} S_n(u)du + C\right), \qquad t \le Z_n \tag{47}$$

where
$$C = \int_{Z_n}^{\infty} \hat{S}(u)du\,.$$

The constant C in the semi-parametric MRL function accounts for the area in the tail of the empirical survivor function that may be lost due to censoring. Note that no attempt is made to estimate the MRL function beyond the last observation. If an estimate is required in this region, the parametric MRL function corresponding to the fitted distribution is used. This technique is extremely flexible in that the constant C is a function of the last observation, Z_n. If Z_n is sufficiently large such that the tail of the empirical survivor function is preserved (or perhaps exaggerated), C will be negligible and the empirical MRL function is not significantly altered. If Z_n is such that significant

area in the tail of the survivor function has been lost, C increases accordingly and the empirical MRL function is improved.

We demonstrate the semi-parametric MRL function technique with an extension of the numerical example presented in Chapter 2. Weibull distributions were fitted to the complete data (Table 4) and censored data (Table 5) to compute the constant C of equation (47) for each case. Leemis (1995) provides details for computing Weibull MLE shape (β) and scale (α) parameters applied here. The shape parameter is found via a Newton-Raphson iterative procedure:

$$\beta_{i+1} = \beta_i - \frac{g(\beta_i)}{g'(\beta_i)} \tag{48}$$

where
$$g(\beta) = \frac{r}{\beta} + \sum_{i \in U} \ln Z_i - \frac{r \sum_{i=1}^{n} Z_i^{\beta} \ln Z_i}{\sum_{i=1}^{n} Z_i^{\beta}};$$

$$g'(\beta) = -\frac{r}{\beta^2} - \frac{r}{\left(\sum_{i=1}^{n} Z_i^{\beta}\right)^2}\left[\left(\sum_{i=1}^{n} Z_i^{\beta}\right)\left(\sum_{i=1}^{n} (\ln Z_i)^2 Z_i^{\beta}\right) - \left(\sum_{i=1}^{n} Z_i^{\beta} \ln Z_i\right)^2\right];$$

r = number of observed failures;

U = set of observed failures.

An initial guess $\beta_0 = 1.5$ was used to start the procedure and a tolerance $\varepsilon = 0.01 > |\beta_{i+1} - \beta_i|$ was used for the final result. Once β is found, the scale parameter is found via

$$\alpha = \left(\frac{r}{\sum_{i=1}^{n} Z_i^{\beta}}\right)^{1/\beta}. \tag{49}$$

The complete and censored data cases of this example are considered separately.

Complete Data. Recall that the tail of the non-parametric survivor function was exaggerated (Figure 5) causing a significant departure of the non-parametric MRL functions from the true distribution (Figure 3). The semi-parametric MRL technique discussed above should show little difference with the non-parametric results since the tail of the survivor function was preserved in the non-parametric result. Application of equation (48) resulted in Weibull parameter estimates of $\beta_n = 1.428$ and $\alpha_n = 0.993$. The parametric estimates are reasonable compared to the true parameters $\beta = 1.5$ and $\alpha = 1.0$. The complete data non-parametric and parametric survivor functions are shown in Figure 11.

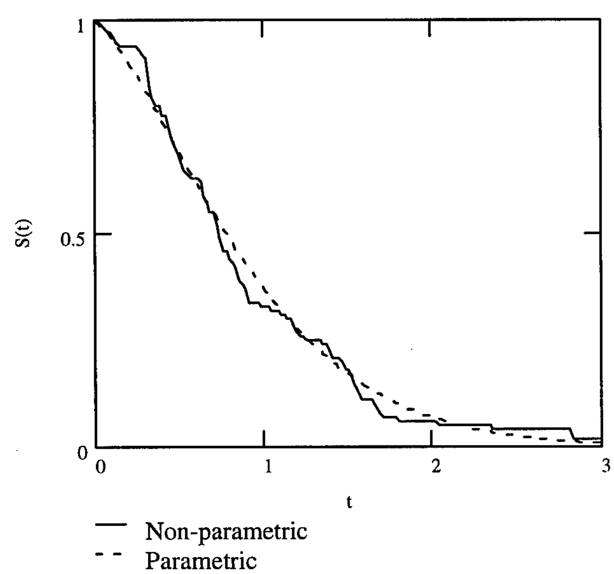


Figure 11. Complete Data Survivor Function Estimates

Note the tail of the parametric estimate is very small past the last observation. In fact, the resulting constant $C = 0.000477$. As desired, the semi-parametric normal and smoothed MRL functions shown in Figure 12 are indistinguishable from the non-parametric functions shown in Figure 3.

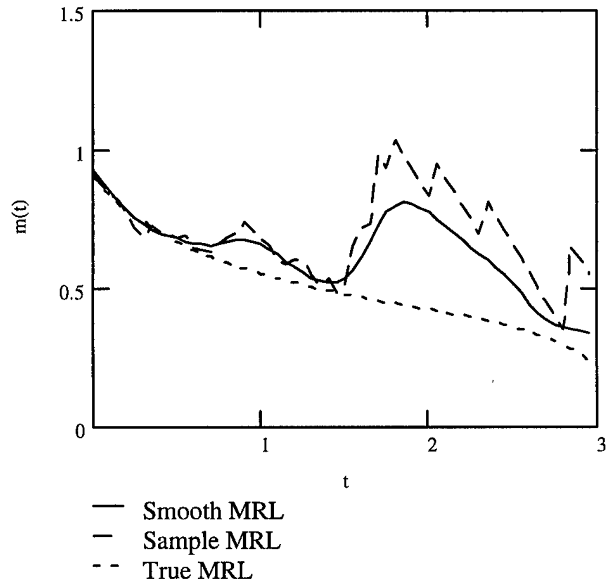


Figure 12. Complete Data Semi-parametric MRL Functions

Censored Data. The three censored data MRL estimation techniques greatly exaggerated the DMRL characteristic of the underlying distribution (Figure 6) due to the truncated nature of the associated non-parametric survivor functions. The semi-parametric MRL technique should alleviate this problem. Application of equation (48) to the censored data set resulted in Weibull parameter estimates of $\beta_n = 1.724$ and $\alpha_n = 1.082$ and the constant from equation (47) $C = 0.011$. The censored data KME survivor function, parametric survivor function, and the true survivor function are shown in Figure 13. The parametric survivor function increases area in the tail of the distribution over that of the non-parametric result, but underestimates the area in the tail of the true survivor function. Still, the semi-parametric MRL functions shown in Figure 14 are a marked improvement over the non-parametric functions (Figure 6). Note the semi-parametric results underestimate the true MRL, but the slope of the DMRL is generally preserved.

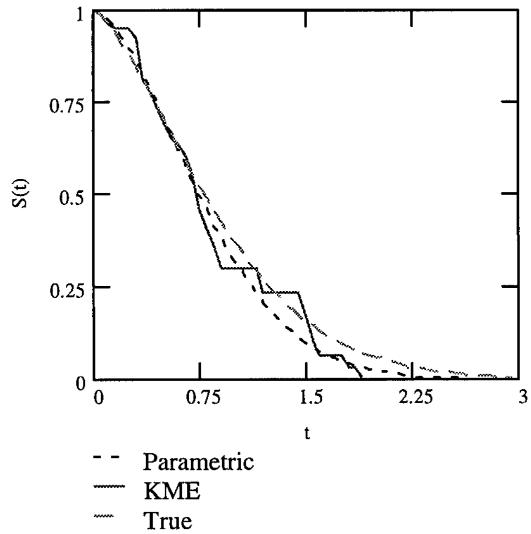


Figure 13. Non-parametric and Parametric Survivor Functions

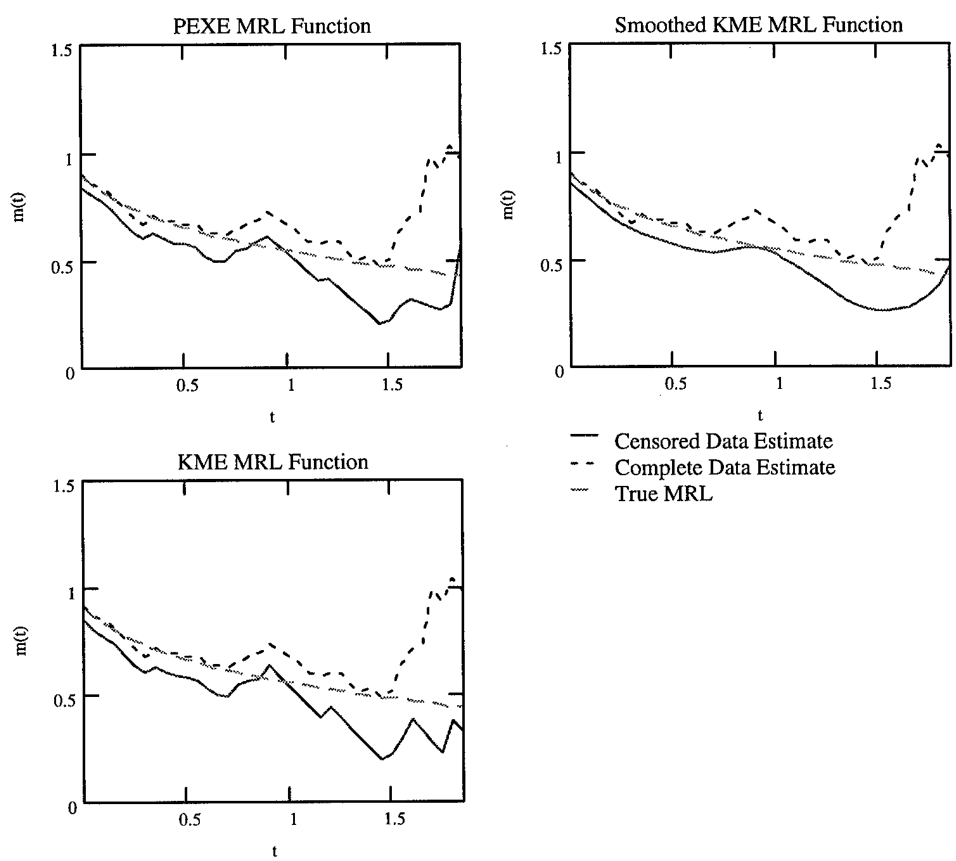


Figure 14. Semi-parametric MRL Functions

MRL Test Performance. The behaviors of the power and level of the tests described in Chapter 2 with respect to sample size, amount of censoring, and MRL characteristic of the underlying distribution must be understood before accepting the implied conclusions of the tests. The tests described in Chapter 2 are examined with respect to these parameters in this section. The complete data and censored data tests are considered separately.

Complete Data. The behaviors of the V^*, Tn, and Un test statistics with changes in the Weibull shape parameter of the underlying true distribution were assessed via a Monte Carlo simulation. (See Appendix B for simulation Mathcad code.) Weibull distribution "data sets" were generated for $\beta = 0.9$ to 1.5 in 0.1 increments with thirty replications performed at each value of β. P-values for each of the three tests were then computed at each replication and β value. The resulting p-value verses β scatter plots are shown in Figure 15. The p-value mean at each β is indicated in the plots to give a sense of the overall trend of the tests as the shape parameter is varied. Plots for n = 100 and n = 200 are shown side by side to capture the effect of sample size on test performance.

Figure 15 suggests the variability of the p-value responses is less for Aly's T_n and Ahmad's U_n tests than for Hollander and Proschan's V^* test. This confirms the claims by Aly and Ahmad that their tests are more efficient than the V^* test. All three tests have a large amount of variability when the underlying distribution is exponential ($\beta = 1$). However, H_0 is incorrectly rejected at the 0.1 level for no more than 10% of the replications in this case. The figure also suggests the intuitive conclusion that the performances of the three tests improve as the sample size increases. Note the variability

of each test is reduced when the sample size is increased to 200 for all values of β except $\beta = 1.0$.

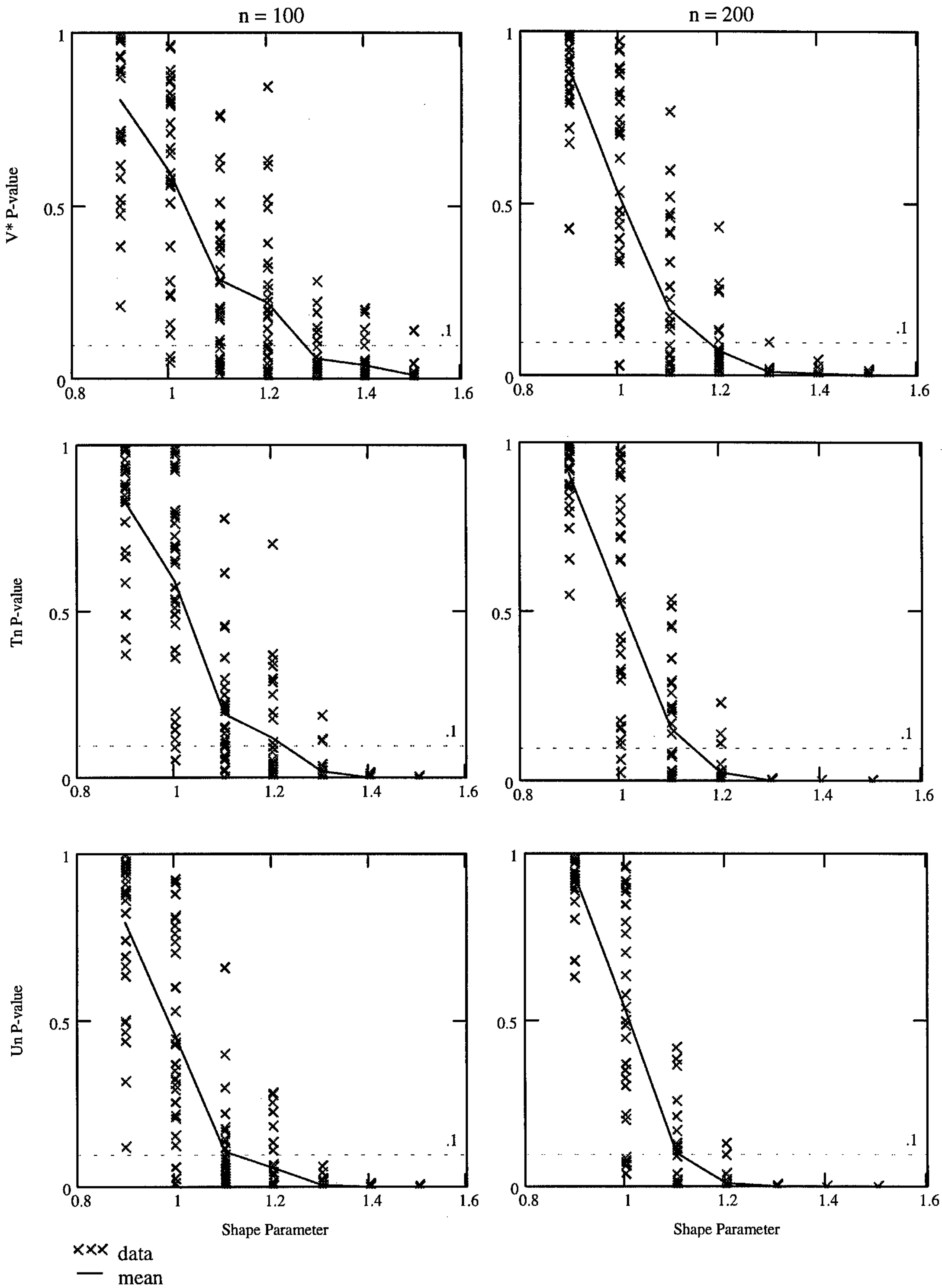


Figure 15. Complete Data Test Performance

Censored Data. The same Monte Carlo simulation procedure used to analyze the complete data tests was applied to the censored data V^c, $L_n^{\ c}$, and $\delta_n^{\ c}$ tests described in Chapter 2. The resulting p-value versus β scatter plots for 45% data censoring are shown in Figure 16. The p-value mean is included in the plots to give a sense of the overall trend of the tests as the shape parameter is varied. Plots for n = 100 and n = 200 are shown side by side to capture the effect of sample size on test performance. The scatter plots for Lim and Koh's $L_n^{\ c}$ test are not included since p-values were << 0.001 for all β values and sample sizes.

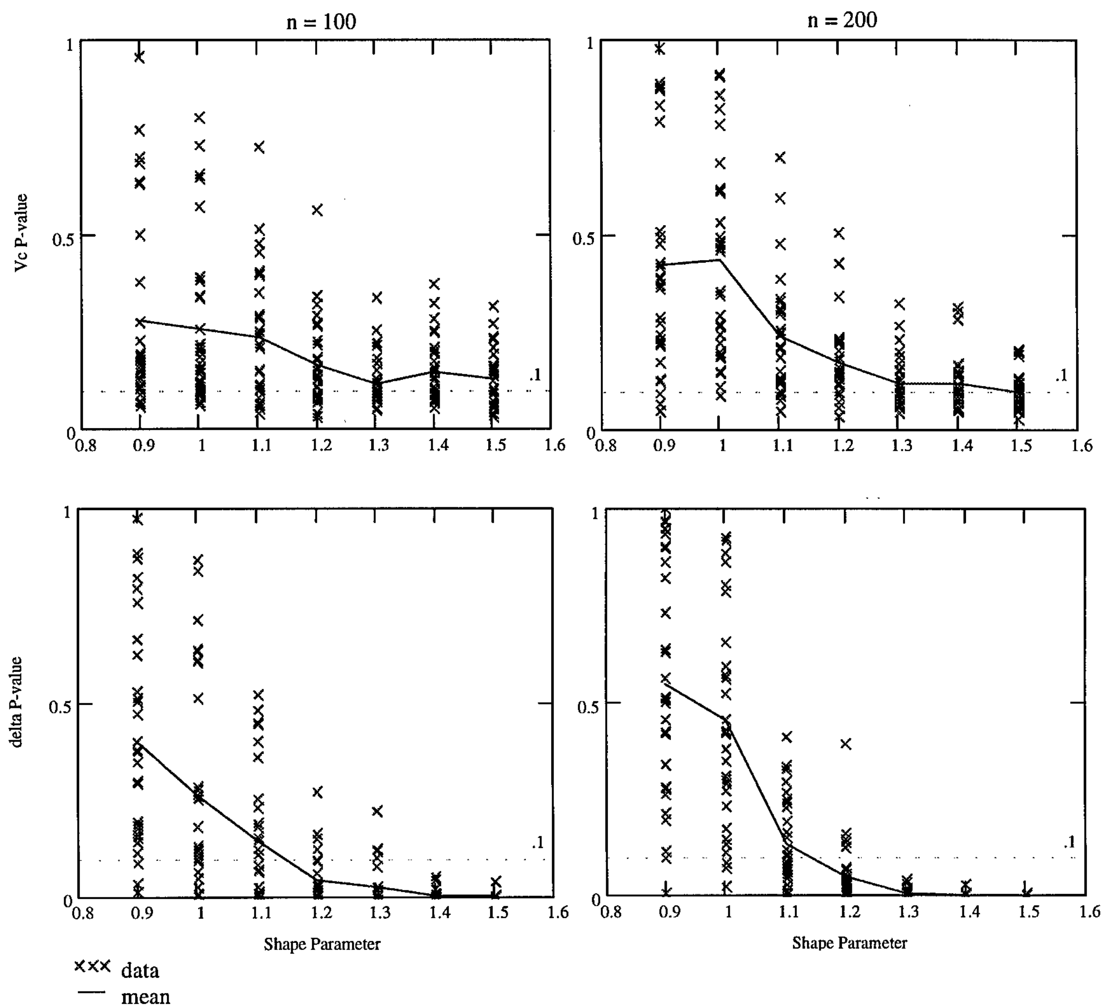


Figure 16. Censored Data Test Performance with Sample Size (45% Censoring)

The overall impression from the scatter plots in Figure 16 is that, as with the complete data tests, the performances of the censored data tests improve with increasing sample size. Furthermore, the figure confirms Lim and Park's assertion that their δ_n^c test is more efficient than Chen Hollander and Langberg's V^c test. Also note the V^c test is unreliable when the data is subject to 45% censoring with a sample size of $n = 100$.

Figure 17 shows p-value versus β scatter plots for $n = 100$ with plots for 9% and 45% censoring shown side by side to capture the effect of censoring percentage on test performance. Again the scatter plots for Lim and Koh's L_n^c test are not included since p-values were << 0.001 for all cases. The tests improve dramatically with decreased censoring. Note the variability in the responses decreases, the probability of rejecting H_o when H_o is false increases, and the probability of rejecting H_o when H_o is true decreases with the lower censoring percentage.

The extremely low p-values obtained in all cases with Lim and Koh's L_n^c are notable. This phenomenon is consistent with the results the authors obtained when they applied their test to failure data consisting of 211 observations subjected to 57% random right censoring. In this example they obtained a test statistic value $n^{1/2}L_n^c/\sigma = 8.75$ with a corresponding p-value << 0.001. Conversely, Chen Hollander and Langberg obtained a test statistic value $n^{1/2}V^c/\tau_0 = 1.52$ with a corresponding p-value = 0.064 and Lim and Park obtained test statistic value $n^{1/2}\delta_n^c/\tau_0 = 3.870$ with a corresponding p-value < 0.001 when their tests were applied to the same data set. In any case, Lim and Koh's L_n^c test was deemed unreliable for this research effort since it never correctly failed to reject H_0 under the conditions of the simulation efforts just presented.

In general, the performance of Chen Hollander and Langberg's and Lim and Park's tests improve as either the sample size increases and/or the amount of censoring decreases. Since it is impractical to attempt quantify the goodness of each test for every sample size and censoring percentage combination, the tests should be subject to simulations similar to the ones presented here before accepting the conclusions of the tests. Recall the current AMRAAM data set consists of 815 observations subject to 74% censoring with future data expected to consist of over 2500 observations subject to 80% censoring. The V^c and $\delta_n^{\ c}$ were subjected to the same Monte Carlo simulation procedure described above to assess their suitability under these conditions with the results shown in Figure 18 and Figure 19 respectively. The figures suggest Chen, Hollander, and Langberg's V^c test is unreliable with these combinations of sample size and censoring percentage. Note the p-value ≈ 0.1 for all β values with no decreasing p-value trend as β increases. Lim and Park's $\delta_n^{\ c}$ test performs better in that there is a decreasing p-value trend with increasing β. However, the figure suggests a high probability of rejecting H_o when H_o is true ($\beta \leq 1.0$).

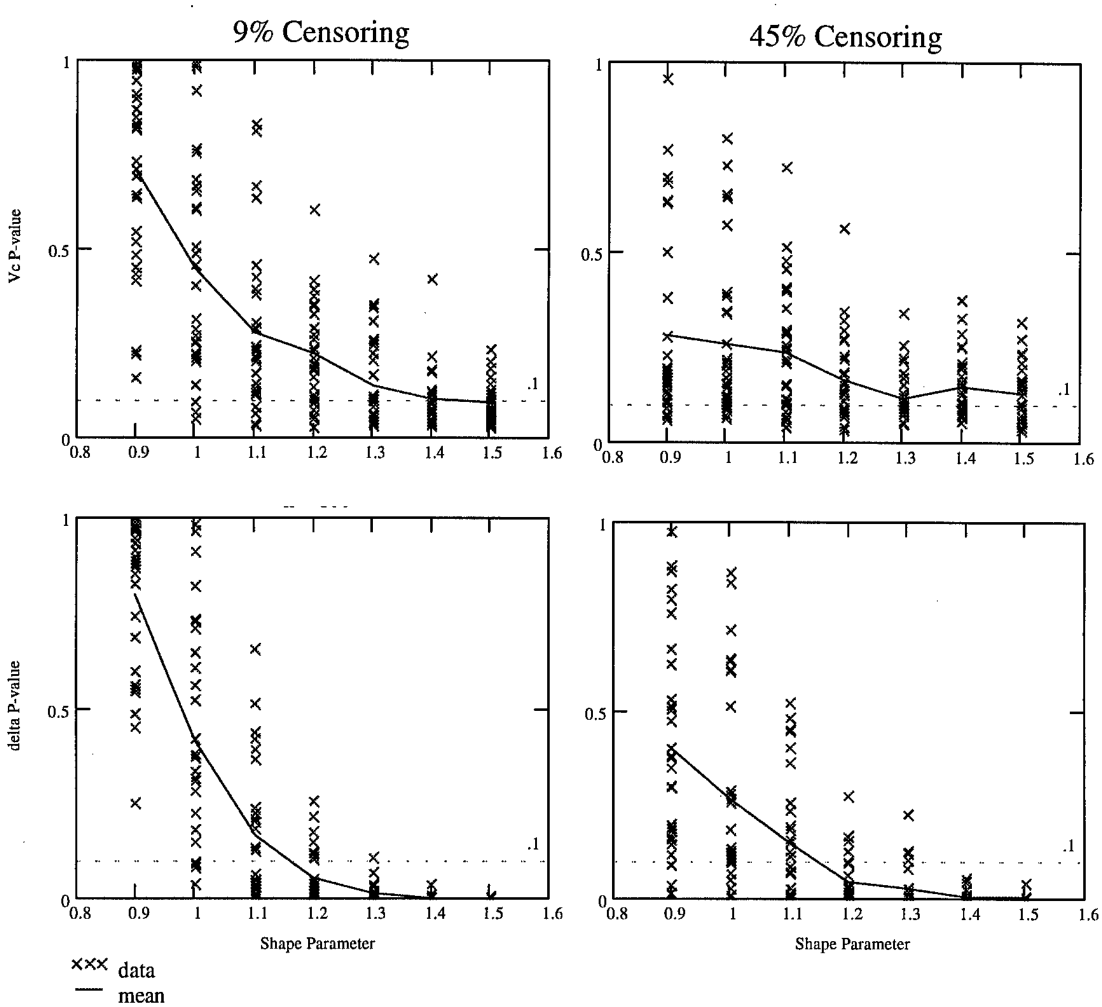


Figure 17. Censored Data Test Performance with Censoring Percentage ($n = 100$)

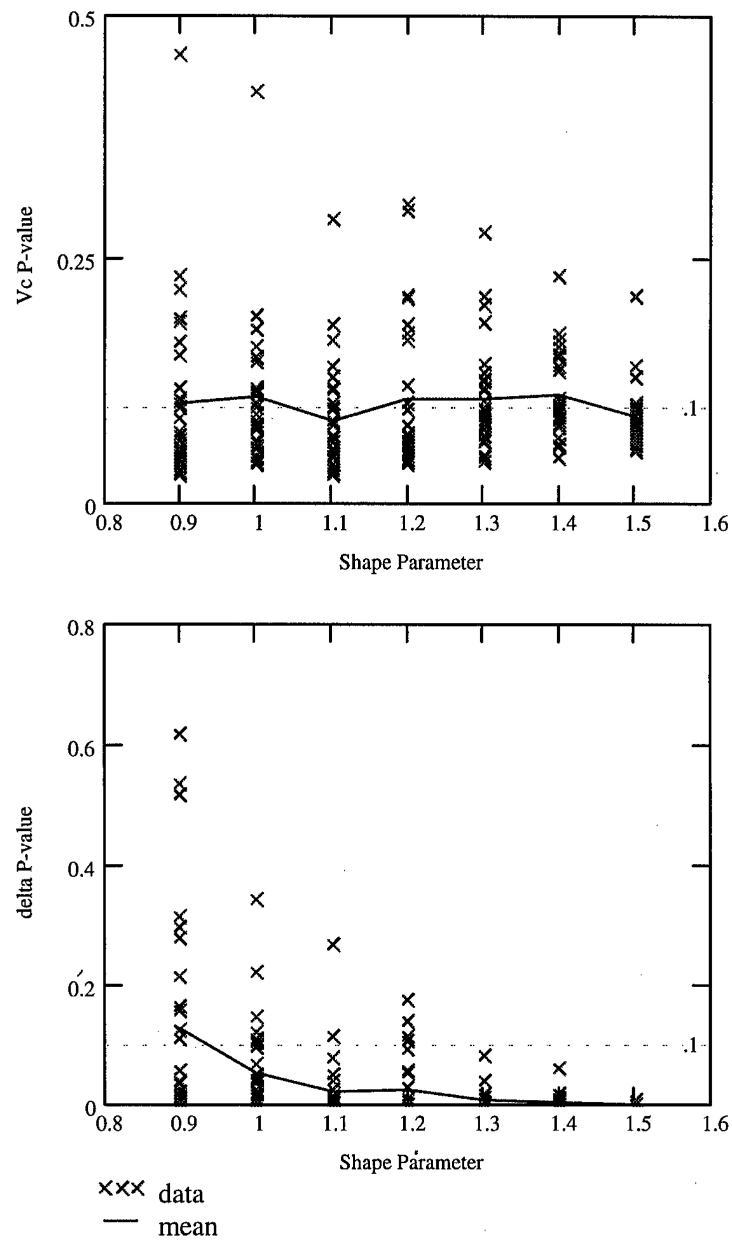


Figure 18. V^c and $\delta_n^{\,c}$ Test Performance with n = 815 and 75.5% Censoring

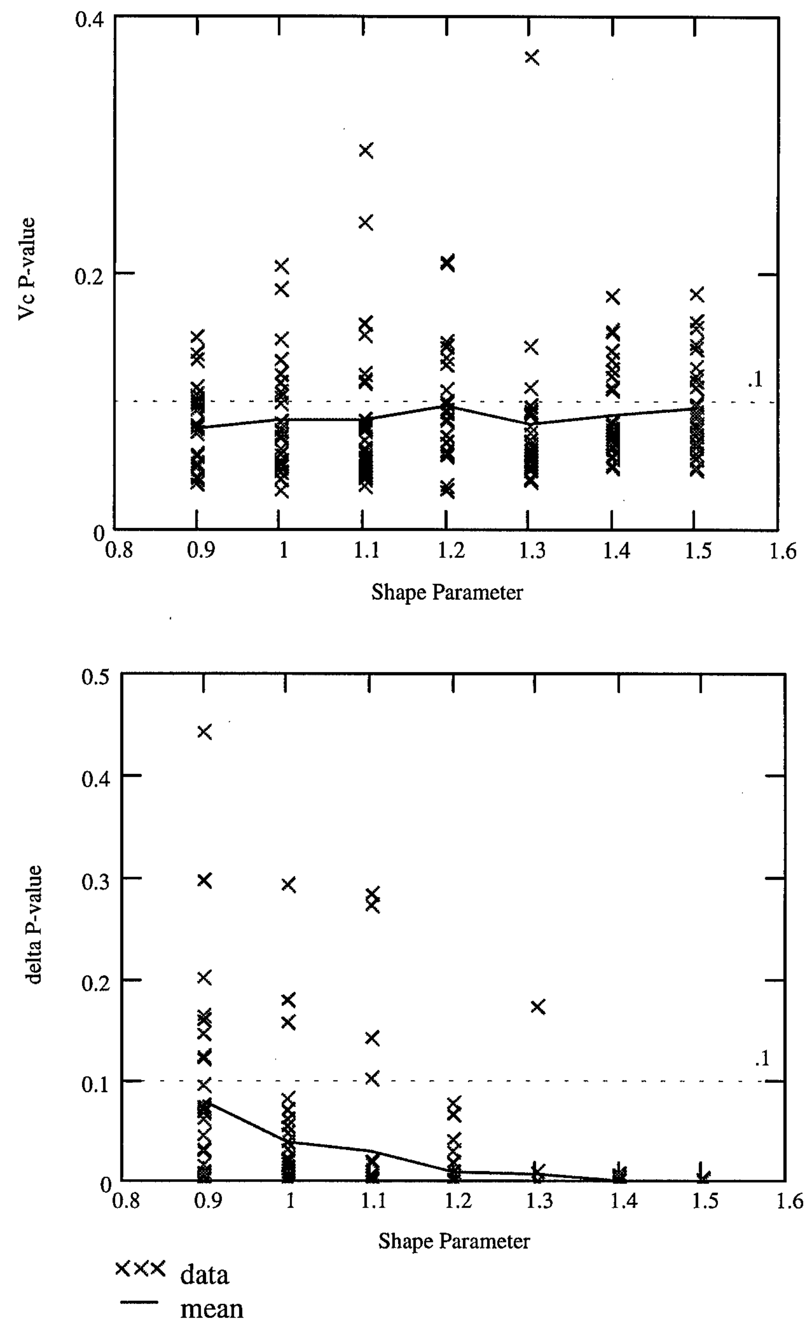


Figure 19. V^c and $\delta_n^{\ c}$ Test Performance with $n = 2500$ and 80.7% Censoring

Preventive Maintenance

Reliability Cost Model Development. As stated in Chapter 2, the traditional approach of maintenance policy optimization through minimizing the cost function can not be directly applied when the penalty of system failure is subjective. In this case, a cost model to express system reliability as a function of cost for each preventive maintenance policy is more appropriate. The cornerstone of this "reliability cost model" is the ability to compute the costs of the respective preventive maintenance policies given a reliability goal. Recall that the age, block, and opportunistic preventive maintenance policies described in Chapter 2 have planned system renewals at the replacement period T. Thus, T is directly related to system reliability. A small value of T results in greater reliability since the system is preventively returned to new condition more often, thereby lowering the probability of failure. Since the respective policy costs are a function of T, a link between system reliability and policy costs is established.

Reliability Determination. The survivor function of an item, $S(t) = 1 - F(t)$, is the probability the lifetime of an item is greater than t and therefore is a measure of the item's reliability at time t. One might suppose that selecting T such that $S(T)$ equals the desired reliability goal is an appropriate method for specifying system reliability with a preventive maintenance policy. However, this intuition is erroneous. Consider the example of an item with an exponential lifetime distribution with associated survivor function as shown in Figure 20. It appears a preventive maintenance policy can be applied with a value of T selected to meet a reliability goal. We know, however, a preventive maintenance policy can not improve reliability since the item is always "as good as new" due to the memoryless property of the exponential distribution.

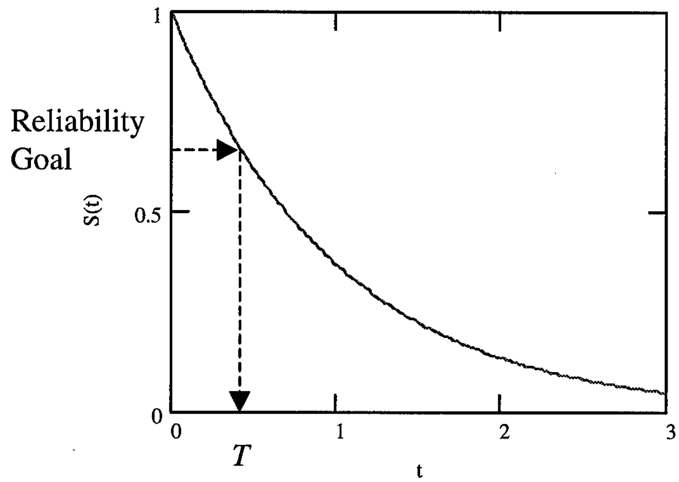


Figure 20. Exponential Distribution Survivor Function

The conflict arises since the survivor function is a measure of projected reliability starting from $t = 0$, and does not consider the probability of survival at time t given the item has survived to some time between 0 and t. The "virtual age" of an item described in Chapter 2 is a measure of this conditional passage of time. Let T_v denote the "virtual replacement period" determined from the survivor function as in Figure 20. From equation (4), T_v can be expressed

$$T_v = m(0) - m(T) . \tag{50}$$

Rearranging we obtain

$$m(T) = m(0) - T_v . \tag{51}$$

Given a value of T_v determined from the survivor function, the actual replacement period T can then be determined from the MRL function. This process applied to the exponential item in the example is shown graphically in Figure 21. Note there is no solution to equation (51) given a value of $T_v > 0$ since $m(t) = m(0)$ is constant for all t. This confirms the notion that a preventive maintenance policy is not suitable for an item with an exponential lifetime distribution.

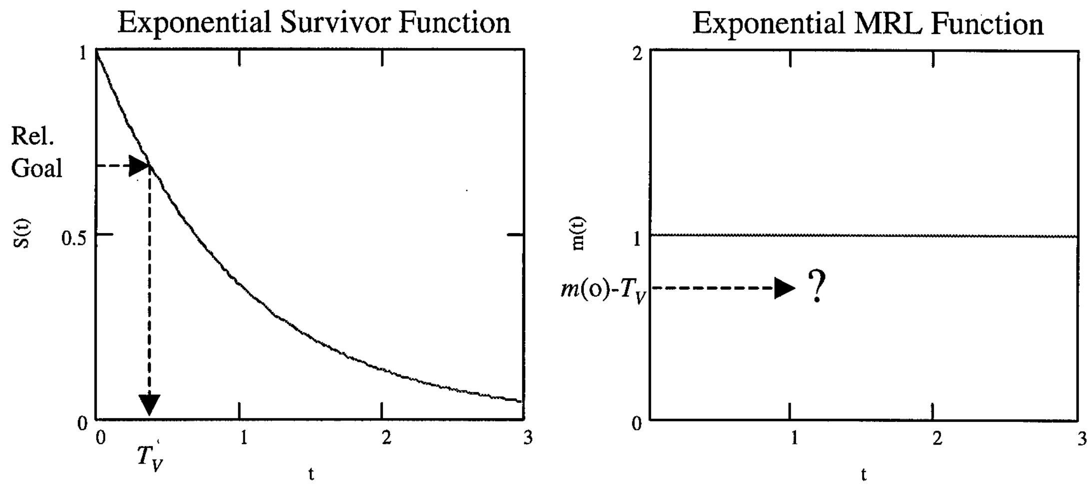
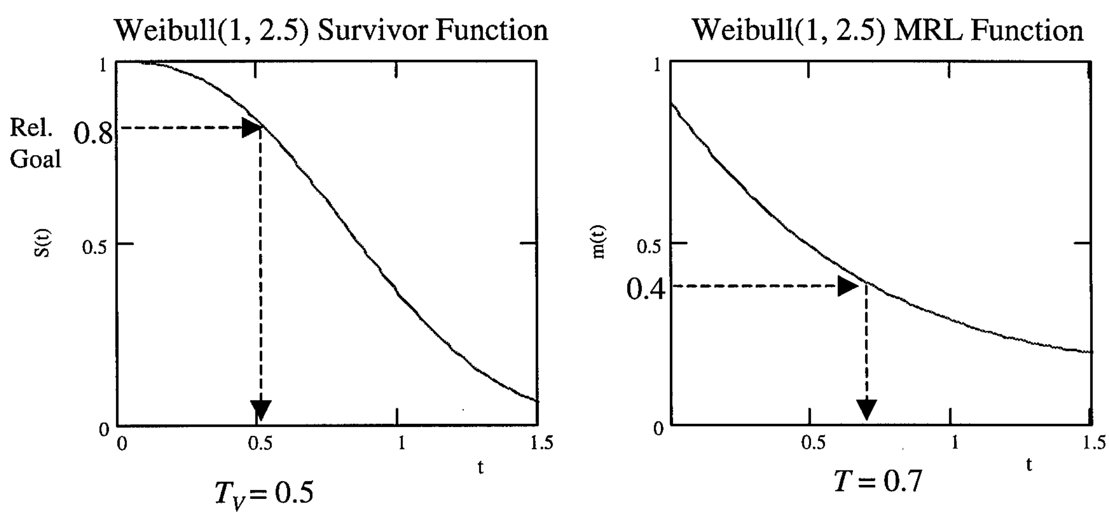


Figure 21. Replacement Period determination with the Exponential Distribution

To illustrate the process with an aging item, consider a Weibull lifetime distribution with survivor and MRL functions as shown in Figure 22. A reliability goal of 0.8 yields a virtual replacement period $T_v = 0.5$. Applying equation (51) we obtain $m(T) = m(0) - T_v = 0.9 - 0.5 = 0.4$. From the MRL function the true replacement period T is determined to be 0.7.



$m(T) = m(0) - T_V = 0.9 - 0.5 = 0.4$

Figure 22. Replacement Period determination with the Weibull (1, 2.5) Distribution

Since it is somewhat awkward to enter the survivor function and MRL function graphs with the respective dependant variables, a slightly modified procedure to apply the reliability cost model is summarized as follows:

1) Specify a replacement period T
2) Compute preventive maintenance policy costs given T
3) Determine the "virtual replacement period", $T_v = m(0) - m(T)$
4) Determine the system reliability = $S(T_v)$.

This procedure reverses the direction of the arrows shown in Figure 22.

Policy Costs. The policy costs discussed in Chapter 2 must be modified for the reliability cost model. Recall the policy costs for the age, block, and opportunistic preventive maintenance policies (equations (35), (40), and (44) respectively) included the cost of a system failure c_f as defined in equation (37). However, the penalty of a system breakdown (c_b) is not explicitly included in the reliability cost model. Therefore the cost of a system failure becomes

$$c_f = c_s + \sum_{i=1}^{nc} c_i \, . \tag{52}$$

Note that c_f is now equivalent to the cost of a planned system replacement (c_p) as defined in equation (38). The reliability model policy costs for the ARP, BRP, and ORP normalized with respect to c_p are then given by

$$C_{Ar}(T) = \frac{1}{\int_0^T S_0(t)dt} \, , \tag{53}$$

$$C_{Br}(T) = \frac{1 + \left(\frac{1}{c_p}\right) \sum_{i=1}^{4} (c_s + c_i) W_i(T)}{T} \, , \tag{54}$$

$$C_{Or}(\tau,T)=\frac{1+\left(\dfrac{1}{c_p}\right)\sum_{i=1}^{4}(c_s+c_i)W_i(\tau)}{\tau+\int_0^{T-\tau}[1-F_V(\tau,u)]du}. \tag{55}$$

Reliability Cost Model Example. In this section the complete reliability cost model is demonstrated with a numerical example. A system composed of four components is considered. The components are assumed to have Weibull lifetime distributions with parameters as specified in Table 6. A setup repair cost $c_s = 0.1$ was chosen arbitrarily and component replacement costs were specified as shown in Table 7. From equation (38) the cost of a complete system replacement is then $c_p = 1.0$.

Table 6. Component Weibull Distribution Parameters

Component	α	β
1	0.0003034	1.2
2	0.0002716	1.3
3	0.0002848	1.4
4	0.0002736	1.5

Table 7. Component Replacement Costs

Component	c_i	$c_{fi} = c_s + c_i$
1	.3	.4
2	.25	.35
3	.2	.3
4	.15	.25

The system survivor function $S_0(t)$ from equation (1) is

$$S_0(t)=\exp\left(-\sum_{i=1}^{4}(\alpha_i t)^{\beta_i}\right) \tag{56}$$

and the resulting system MRL function $m_0(t)$ is given by

$$m_0(t)=\frac{1}{S_0(t)}\int_t^{\infty}S_0(u)du. \tag{57}$$

Plots of the system survivor and MRL functions are shown in Figure 23.

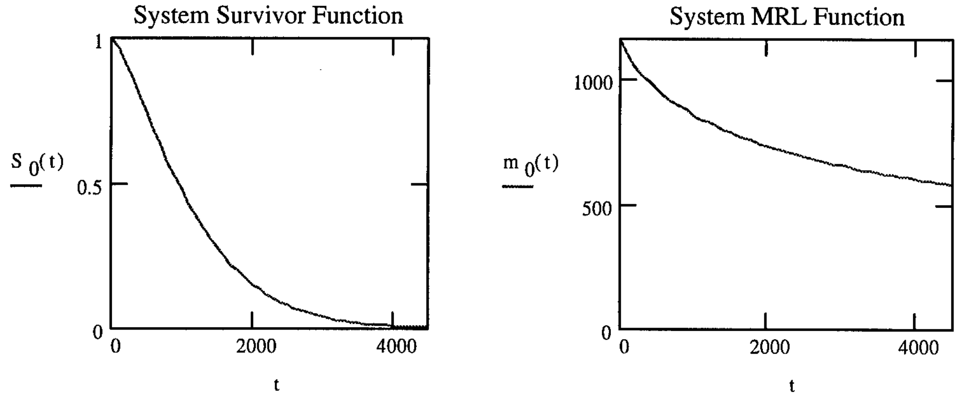


Figure 23. System Survivor and MRL Functions

The reliability cost model summarized above was implemented with the replacement period T varied from 300 to 4500 hours in 100 hour increments. Equation (42) was used to solve the expression in the numerator of the BRP (54) and ORP (55) cost equations. Equation (45) was used to solve the expression in the denominator of the ORP cost equation (55). An iterative procedure whereby the parameter τ was varied from 0 to T in 100 hour increments was used to optimize the ORP (55) with respect to τ at each value of T. All computations and plots were computed via Mathsoft's Mathcad 6.0 software on a PC (Appendix C). Figure 24 shows the resulting policy costs as a function of the computed reliability at each value of T. Figure 25 shows the parameters T and τ plotted against the computed reliability.

Some important observations can be made from this example. First, the policy costs are increasing functions with respect to the reliability goal, and the increases are dramatic when the specified reliability goal exceeds 0.85. Second, the most cost efficient policy (in this case the ORP) remains the same regardless of the specified reliability goal. Third, the distinction between the costs of the different policies diminishes as the

specified reliability goal increases. These observations are consistent with Spearman's results for the traditional cost optimization model employed with several different cost coefficients and Weibull distribution parameters.

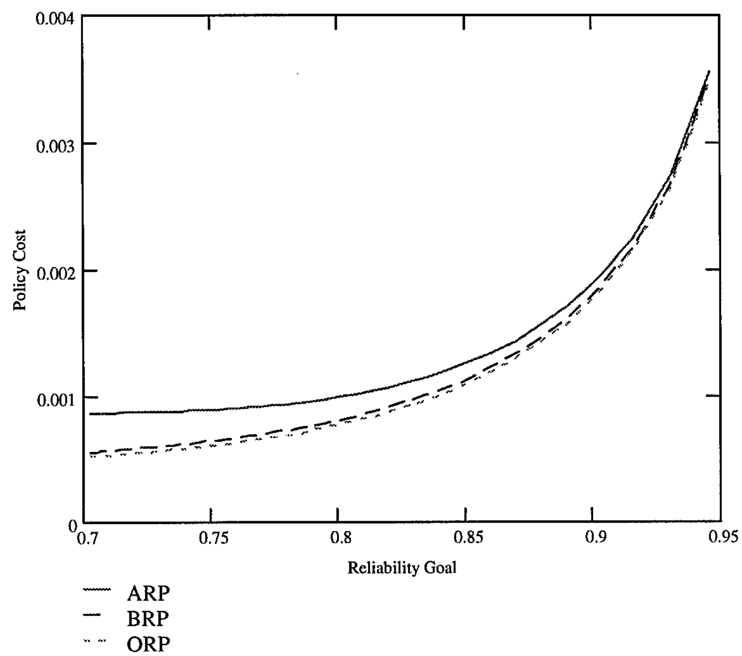


Figure 24. Policy Costs vs. Reliability Goal

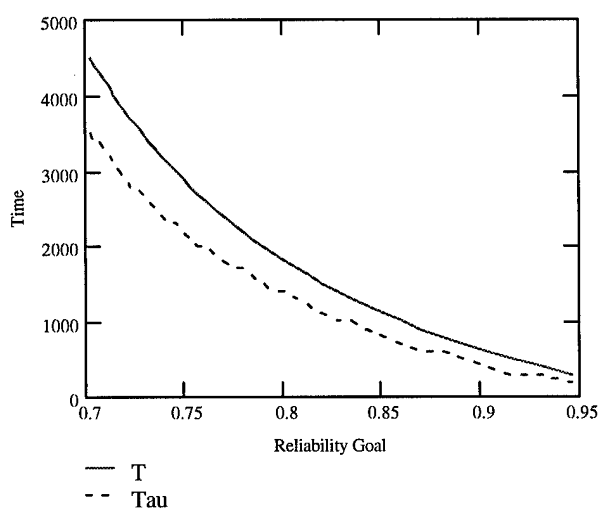


Figure 25. T and τ vs. Reliability Goal

Cost Parameter Sensitivity Analysis.

Spearman's results differed from those presented in the example above in that he found the ARP to be most cost efficient policy. He also found the ORP to be degenerate ($\tau = 0$) with the ARP. The disparities are accounted for by the choice of the various cost coefficients, notably the value of the setup repair cost (c_s) relative to the sum cost of the individual components ($\sum c_i$). If c_s is very large, as in Spearman's examples, the ARP is attractive since the cost c_s is incurred less often than with the BRP. If c_s is relatively small, as in the example presented above, the BRP is attractive since good components are replaced less often than with the ARP.

At issue is the value of c_s that represents the boundary between the ARP and the BRP costs. Figure 26 shows a plot of the three policy costs verses the cost ratio $c_s/\sum c_i$. The parameter c_s was varied from 0 to 4.5 while the other costs were held constant as described in Table 7 to obtain a range of the cost ratio $c_s/\sum c_i$ from 0 to 5.0. Note the boundary between the ARP and the BRP is $c_s/\sum c_i = 0.75$. Furthermore, note that the ORP is always at least as cost efficient as either the ARP or the BRP. As the ratio $c_s/\sum c_i$ becomes very large the optimal value of τ decreases and the ORP becomes degenerate with the ARP as shown in Figure 27. As the ratio $c_s/\sum c_i$ decreases to zero the optimal value of τ approaches T and the ORP becomes degenerate with the BRP. The ORP offers the greatest advantage in cost efficiency when the cost ratio $c_s/\sum c_i = 0.75$.

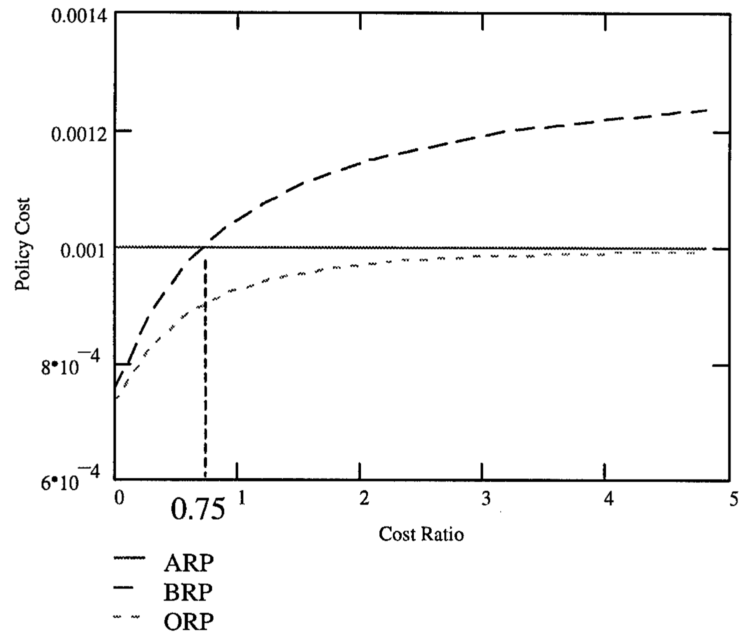


Figure 26. Policy Costs vs. $c_s/\sum c_i$ Cost Ratio, $T = 1800$

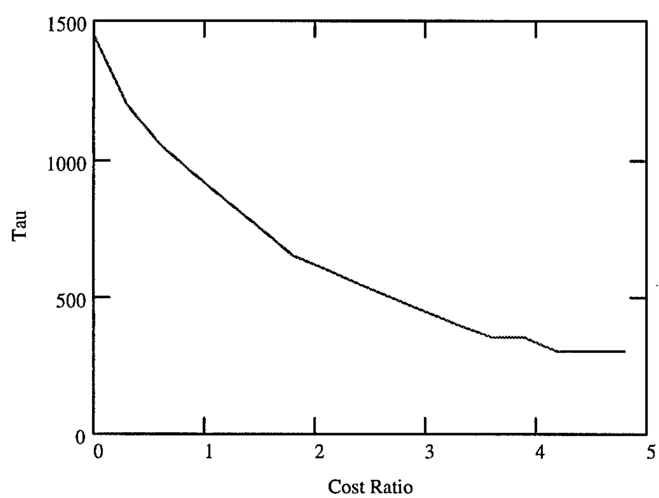


Figure 27. ORP τ vs. $c_s/\sum c_i$ cost ratio

Empirical Reliability Cost Model. The performance of the reliability cost model applied to empirical data was assessed via a Monte Carlo simulation (Appendix C). Weibull distribution "data sets" were generated for a four component series system in accordance with the parameters in Table 6. The data set was censored in accordance with

the following procedure. Let X_{ij} represent the i^{th} failure time for the j^{th} component, then the censored system level failure data is

$$Z_i = \min\{X_{i1}, X_{i2}, X_{i3}, X_{i4}, Y_i\} \tag{58}$$

where Y_i is an observation from the censoring distribution G.

I used an exponential censoring distribution with rate $\lambda = 0.0018$ to obtain an average of 74% random right censoring of the system level data set. Note that if a system level observation is censored ($Z_i = Y_i$), then the observations for each of the components are also censored. If a system level observation is an observed failure ($Z_i \geq Y_i$), then all but the minimum component failure time are censored. Therefore, component level data is always subject to greater censoring than system level data with this censoring scheme.

Reliability Determination. The system reliability $S(T_v)$ is a random variable in the empirical model since it is a function of values calculated empirical estimates of the survivor and MRL functions. To assess the performance of this random variable, 30 replications of $n = 815$ data points were generated as stated above. The system reliability was computed for values of T from 500 to 2000 hours in 300 hour increments for each replication. This procedure was repeated for each survivor and MRL function estimation technique: KME, PEXE, and smoothed KME respectively.

Figure 28 shows the resulting reliability verses T scatter plot using the semi-parametric KME estimate for the MRL function. The mean and its associated 95% confidence interval using the student's t statistic are also shown to provide a sense of the overall trend and variability of the responses. The true reliability curve is shown for comparison. The 95% confidence interval covers the true result for all values of T considered (except $T = 1700$) suggesting no statistical difference between the empirical

mean and the true result. The same simulation was performed using the non-parametric KME MRL function with the results shown in Figure 29. The random number seeds used for the previous simulation were repeated, therefore the differences between the two results are attributed solely to the different MRL estimation techniques. Note that the reliability for a given value of *T* is generally underestimated when the non-parametric MRL function is used. The 95% confidence interval does not cover the true result indicating statistical significance between the empirical and true result. This characteristic is attributed to the generally exaggerated DMRL functions of this technique.

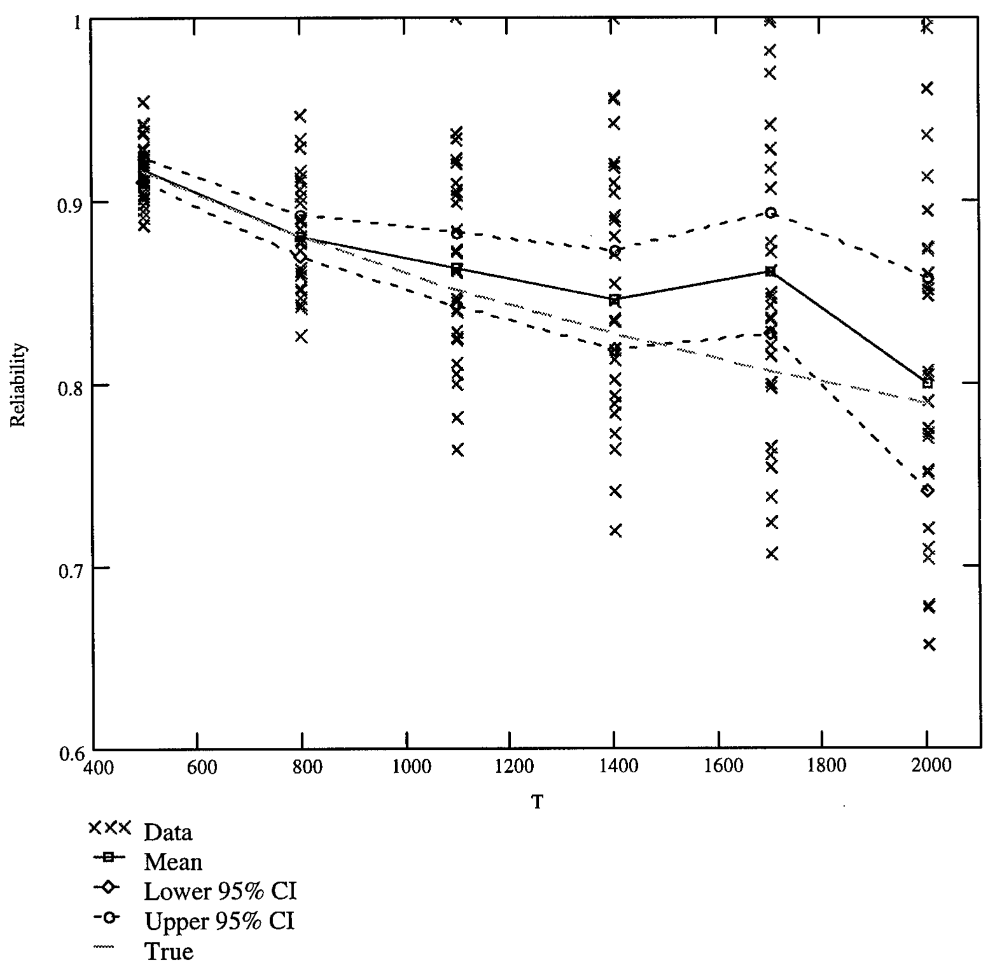


Figure 28. Semi-parametric KME Reliability vs. *T*

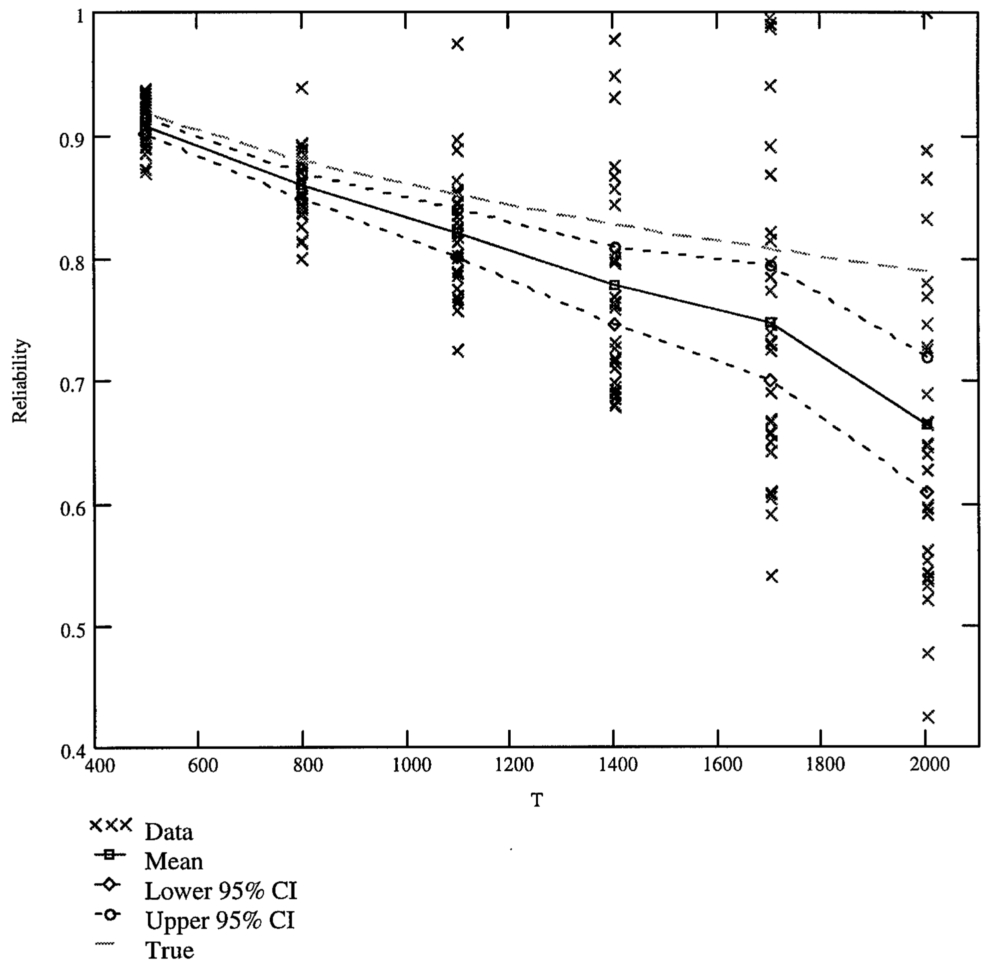


Figure 29. Non-parametric KME Reliability vs. T

Although the semi-parametric KME result performed well on average, the variability in the responses is of concern, especially for large values of T. Figure 30 and Figure 31 show similar reliability verses T scatter plots with using the semi-parametric PEXE and smoothed KME (h = 100) MRL functions respectfully. The random number seeds used for the KME simulations were repeated allowing direct comparison of the results. The PEXE MRL function results are similar to those for the semi-parametric KME. Although a slight improvement in the mean response for the larger values of T is detectable, there is no significant reduction in the variability of the responses.

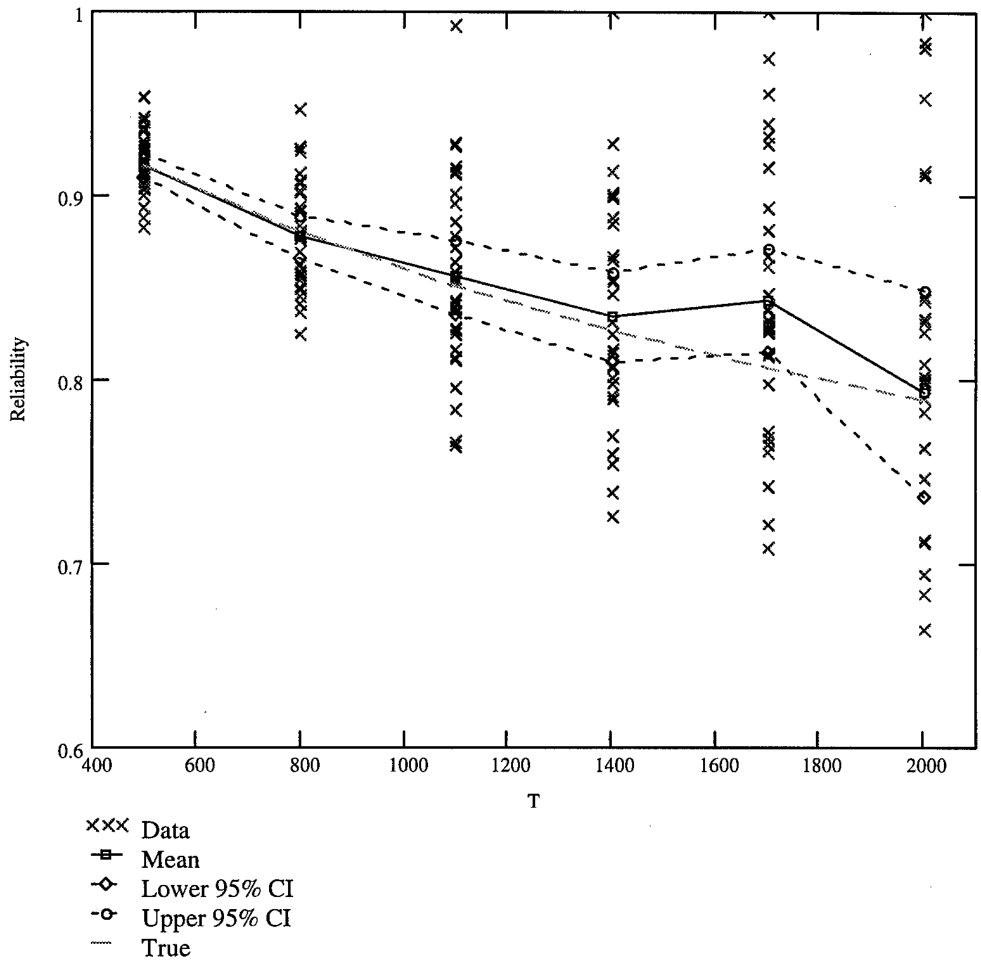


Figure 30. Semi-parametric PEXE Reliability vs. T

The smoothed KME MRL function ($h = 100$) provided a slight reduction in the variability of the responses over those of the semi-parametric KME and PEXE MRL functions. A smoothing parameter $h = 200$ was attempted with the results shown in Figure 32. Although the variability of the responses is reduced over the previous cases, the reliability is underestimated for small values of T.

In summary, the semi-parametric smoothed KME MRL function provides the best results for reliability determination of the three MRL estimation techniques. However, the smoothing parameter h must be chosen judiciously. The smoothing parameter should be chosen as large as possible without so distorting the empirical survivor and MRL

functions that an accurate reliability determination is not possible. Also, regardless of the empirical survivor / MRL function estimation technique, the Monte Carlo simulations show an accurate estimation of reliability can not be assumed when T is large and approaches the value of the last observation. This is not considered to be a significant limitation, however, since in most cases where a preventive maintenance policy is considered small values of T are desired.

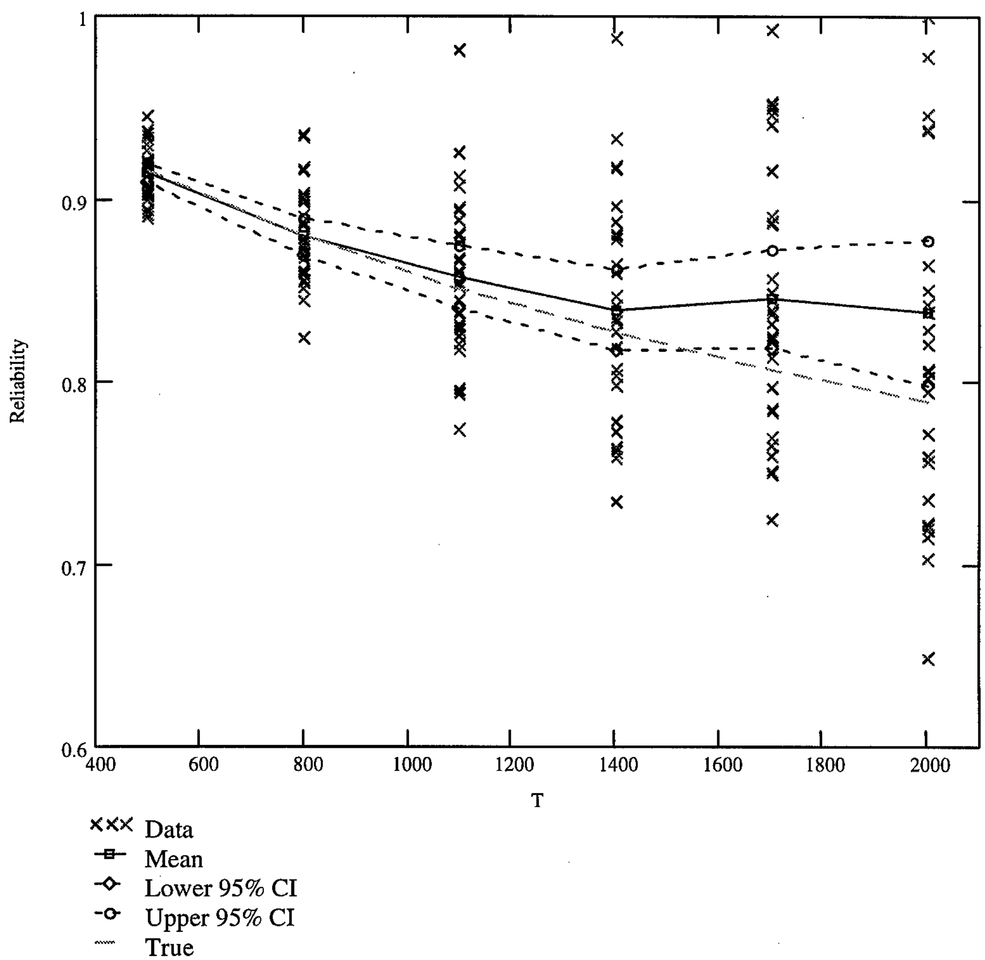


Figure 31. Semi-parametric Smoothed KME (h = 100) Reliability vs. T

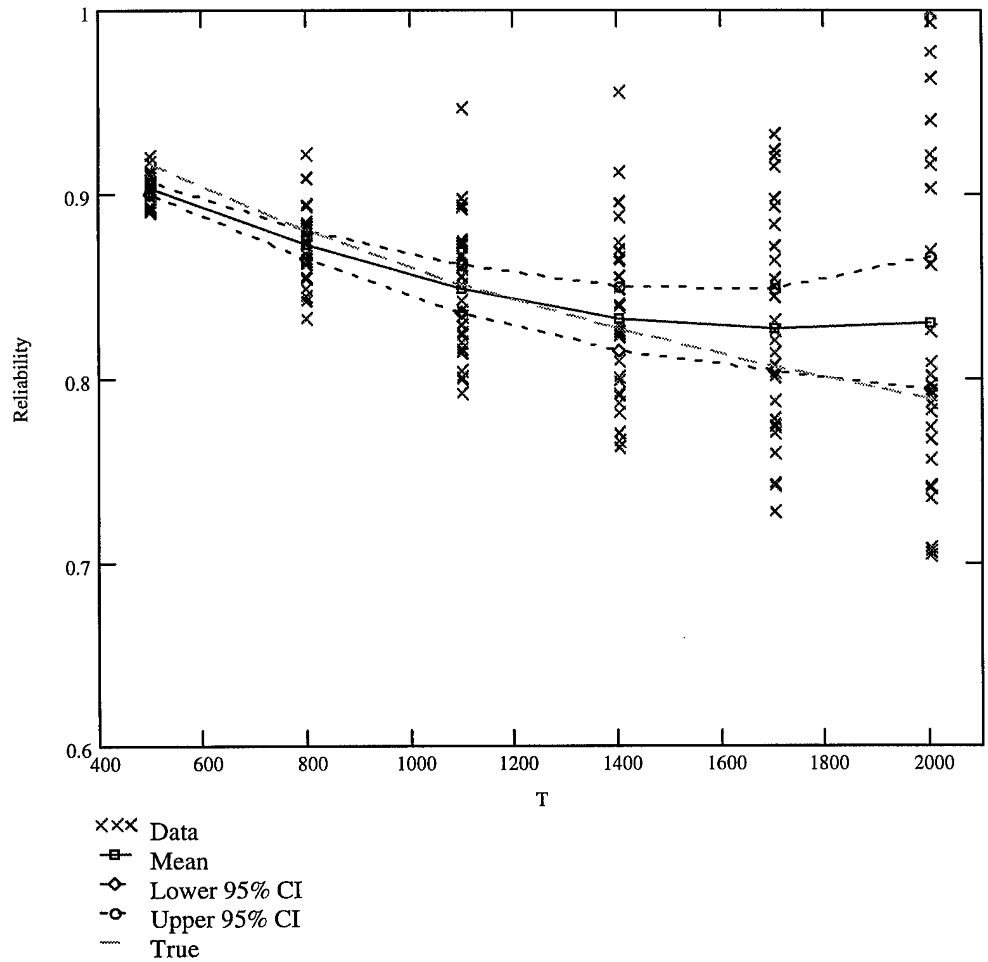


Figure 32. Semi-parametric Smoothed KME ($h = 200$) Reliability vs. T

Policy Costs. The costs of the three preventive maintenance policies are also random variables in the empirical model since their calculation requires estimates of either the system (ARP) or component (BRP and ORP) lifetime distribution functions. The performances of the empirical policy costs were assessed with the same Monte Carlo simulation procedure described above for the reliability determination. The empirical ARP cost was determined via equation (53) with KME of the system survivor function used in place of $S_0(t)$. The empirical BRP cost was determined via equation (54). The component renewal functions, $W_i(t)$, were solved via equation (42). Weibull MLE

parameters were determined via equation (48) and used to compute parametric estimates for μ, σ^2, $f(t)$, and $F(t)$ using the equations presented in Table 2. The empirical ORP cost was determined via equation (55). The component renewal functions were computed as described above and the expression in the denominator was solved using equation (45) with MLE estimates used for the component Weibull parameters.

Figure 33, Figure 34, and Figure 35 show the resulting cost verses T scatter plots for the ARP, BRP, and ORP respectively. The respective true cost functions are shown for comparison. The figures suggest the empirical cost models very accurately reflect the theoretical results with very little variability. Figure 36 shows the ORP optimal value of τ versus T scatter plot. Again the empirical model performed very well compared to the theoretical result.

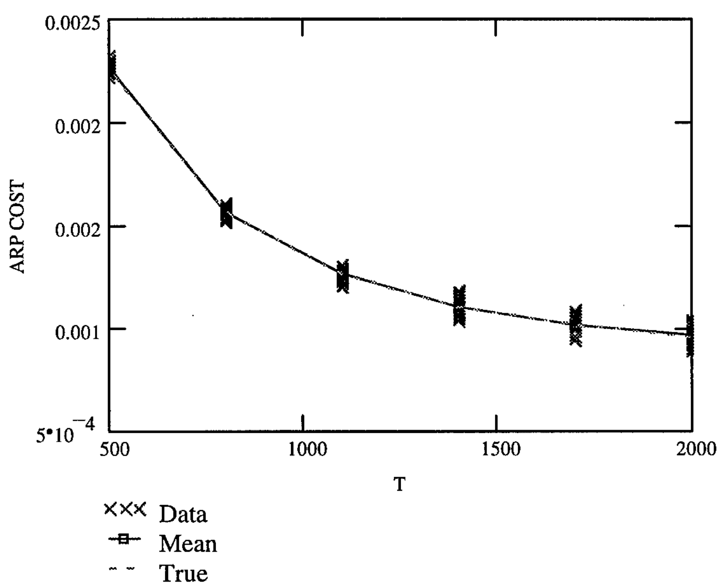


Figure 33. ARP Cost vs. T

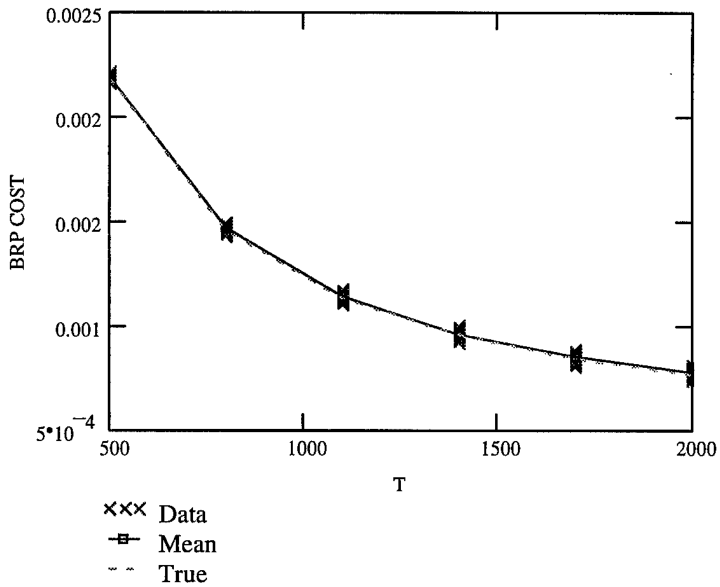


Figure 34. BRP Cost vs. *T*

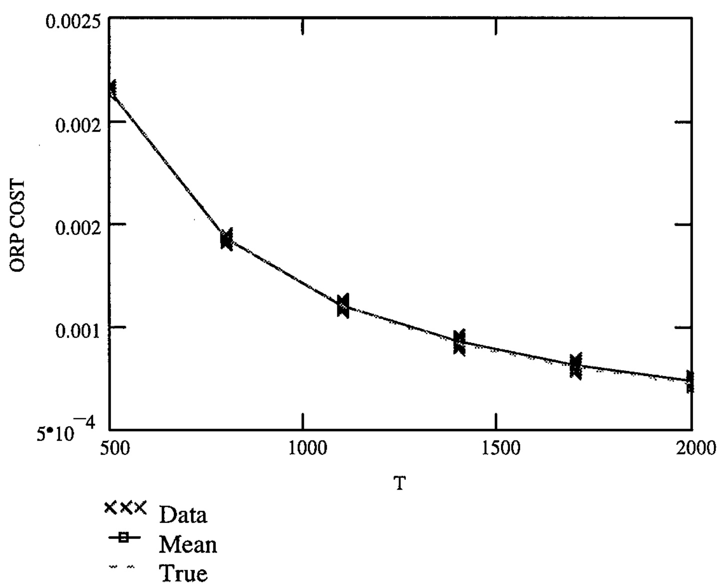


Figure 35. ORP Cost vs. *T*

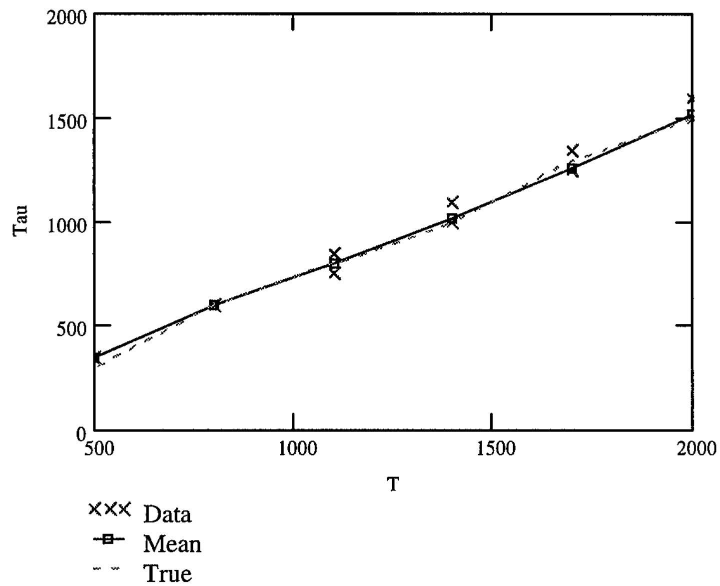


Figure 36. ORP τ vs. T

IV. AMRAAM Data Analysis

Overview

In this chapter the AMRAAM failure data is analyzed with the methods developed in Chapter 3. As of this writing, the AMRAAM failure data consists of $n = 815$ total observations, 606 of which are censored resulting in 74.4% random right censoring. The data set consists of system failure data only. Lacking component level data, the ARP was the only preventive maintenance policy that could be considered for analysis.

The chapter begins with an analysis of the empirical MRL function. The empirical survivor functions are developed and the non-parametric, parametric, semi-parametric MRL functions are examined. The chapter concludes with an application of the reliability cost model developed in Chapter 3.

MRL Analysis

Survivor Function Estimates. Figure 37 shows the PEXE, Smoothed KME (h = 100) and KME survivor function estimates. All three non-parametric estimation techniques match each other very closely. Of note is the relative "smoothness" of the KME survivor function. The very small "steps" in the KME survivor function are due to the compactness of the data for $t \leq 1600$. Most startling, however, is the obvious truncated nature of all three of the non-parametric survivor function estimates. The survivor function estimates abruptly end at $t = Z_n = 2070$ with $S(Z_n) \approx 0.3$. A Weibull distribution was fitted to the data with MLE parameters $\alpha_n = 0.000616$ and $\beta_n = 0.853$ determined via equation (48). The Weibull parametric survivor function estimate is compared to the KME estimate in Figure 38. The non-parametric estimate is a close

match to the parametric result for $t < Z_n$, but the lack of information in the tail of the non-parametric distribution is evident.

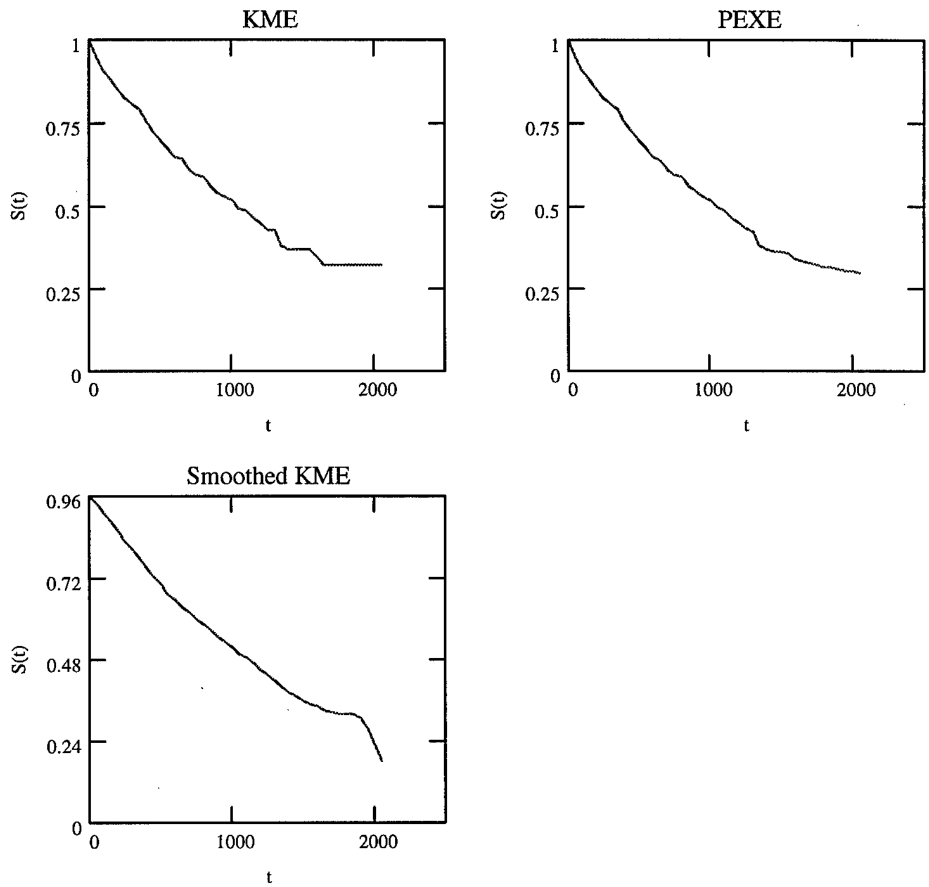


Figure 37. Non-Parametric Survivor Function Estimates

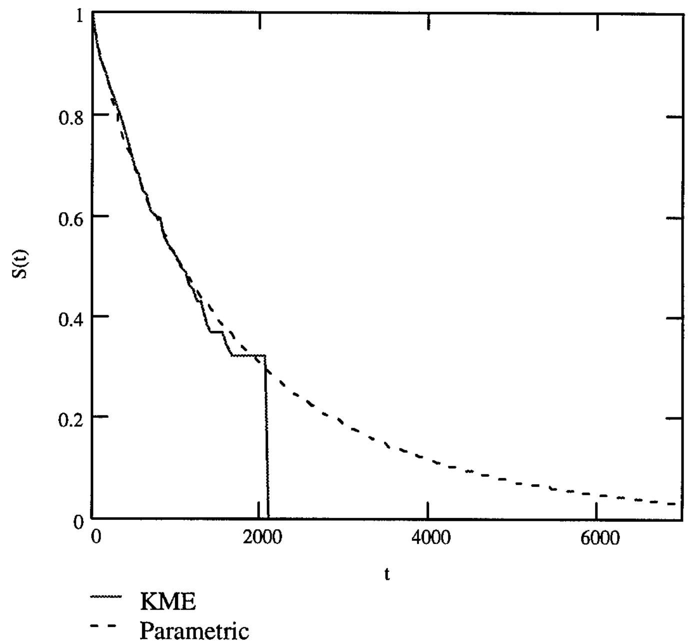


Figure 38. KME and Parametric Survivor Function Estimates

MRL Function Estimates. Figure 39 shows the KME, semi-parametric KME, and Weibull parametric estimates of the MRL function. The smoothed KME and PEXE MRL estimates are time intensive to compute and were not pursued due to the already relatively smooth nature of the KME MRL function. The figure illustrates the dramatic departure of the non-parametric KME MRL function from the parametric and semi-parametric results. The decreasing MRL nature of the non-parametric KME result is most certainly exaggerated since the non-parametric KME survivor function contains no information in the tail of the underlying distribution. The semi-parametric and parametric results are in close agreement and indicate a slightly increasing MRL function. However, these results may also differ significantly from the true underlying distribution since such a large portion of the survivor function is estimated beyond the data as shown in Figure 38.

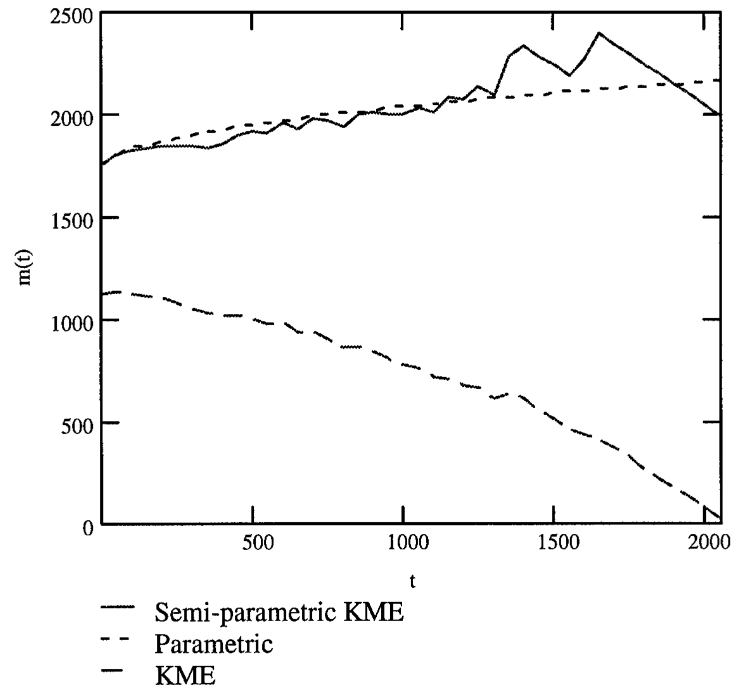


Figure 39. KME, Semi-Parametric KME, and Parametric MRL Functions

MRL Statistical Tests. Chen Hollander and Langberg's V^c test and Lim and Park's $\delta_n{}^c$ test were applied to the data to test for DMRL and NBUE respectively. The V^c test resulted in a test statistic value of 1.967 and a corresponding p-value = 0.025. The $\delta_n{}^c$ resulted in a test statistic value of 5.855 with a corresponding p-value << 0.001. Both tests confirm the DMRL / NBUE impression of the KME MRL function. However, these tests are unreliable under the sample size and censoring percentage conditions of the AMRAAM data set as demonstrated in Figure 18. Based on the impression of the parametric and semi-parametric MRL functions, I would not reject H_0 in favor of DMRL or NBUE under these conditions.

Preventive Maintenance Policies

The aging criterion must be satisfied before a preventive maintenance policy is pursued. The MRL analysis presented above does not support, nor necessarily reject, this

criterion. Without conclusive evidence of aging, the pursuit of a preventive maintenance policy for the AMRAAM based on the current data is not warranted. However, for illustration purposes the reliability cost model developed in Chapter 3 is applied here using the non-parametric MRL function.

Reliability Determination. The system reliability $S(T_v)$ was computed for values of T from 500 to 2000 hours in 200 hour increments using the KME survivor and MRL functions. Figure 40 shows the resulting reliability versus T plot.

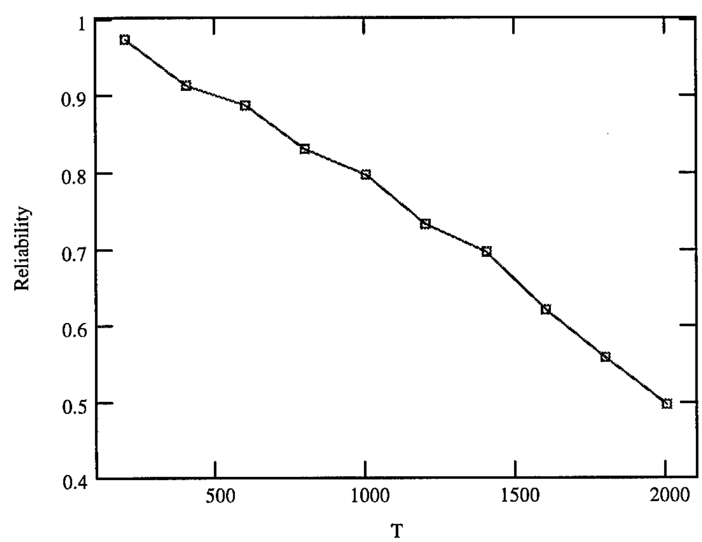


Figure 40. Reliability vs. T

ARP Cost. The normalized ARP cost given by equation (53) was computed for values of T from 500 to 2000 hours in 200 hour increments. Figure 41 shows the resulting ARP cost versus T plot. Note the dramatic increase in cost for $T < 500$ which represents a desired reliability ≥ 0.9 as shown in Figure 40. Figure 42 shows the direct relationship of the ARP cost with the desired reliability goal.

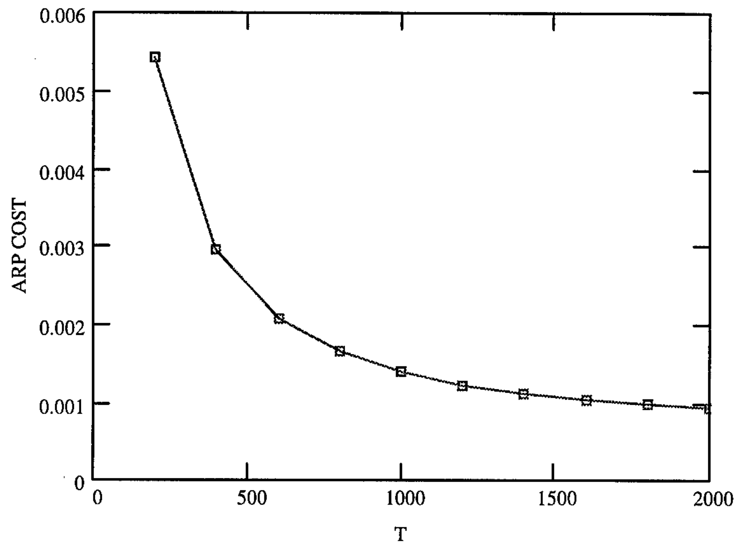


Figure 41. ARP Cost vs. *T*

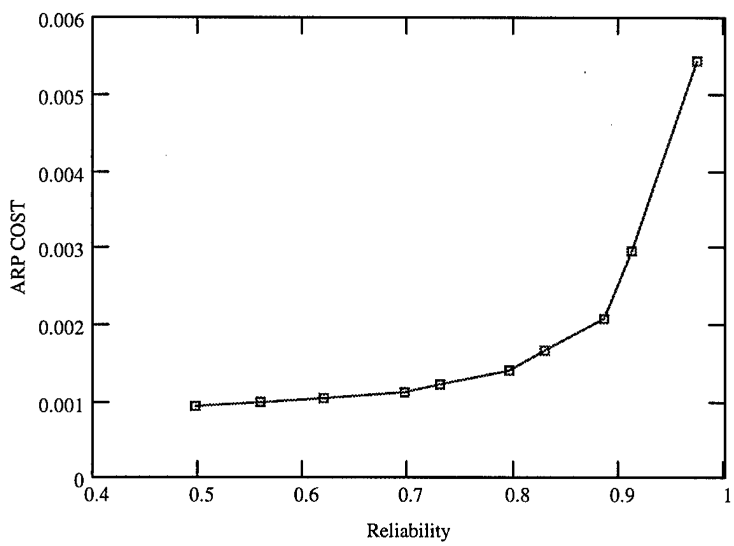


Figure 42. ARP Cost vs. Reliability

V. Summary and Conclusions

Research Objectives and Primary Results

The overall objective of this research effort was to formulate a preventive maintenance strategy for using, retiring, or refurbishing AMRAAM missiles subject to extended captive carry flight time. A preventive maintenance policy is only applicable if the item in question is aging, or deteriorating with time. Therefore, a supporting objective of this research is to characterize the aging process of the missile system through a non-parametric analysis of its Mean Residual Life (MRL) function.

The MRL analysis of the current AMRAAM failure data did not support the aging criterion. Without conclusive evidence of aging, the pursuit of a preventive maintenance policy for the AMRAAM based on the current data is not warranted. Instead a failure replacement maintenance policy whereby individual components are repaired / replaced upon failure is recommended.

Summary of Other Significant Results

Non-parametric MRL Functions. Three non-parametric MRL function estimation techniques discussed in the literature were examined via a numerical example. All three estimation techniques differed significantly from the true MRL function in that they exhibited a greatly exaggerated decreasing MRL. The departure from the true result was attributed to the lack of information in the tail of the respective non-parametric survivor functions. A semi-parametric technique for estimating the MRL function was developed whereby the tails of the respective non-parametric survivor functions were estimated with a parametric result.

The semi-parametric MRL functions showed dramatic improvement over the non-parametric estimates.

MRL Statistical Tests. The performances of several statistical tests for the distribution classes of DMRL and NBUE were assessed via Monte Carlo simulation. Complete data and censored data tests were considered. The simulations revealed the behavior of the tests with changes in sample size, amount of censoring, and the MRL characteristic of the underlying distribution. The simulations also showed that none of the three censored data tests considered were reliable under the conditions of 1) sample size of 815 with 75% random right censoring, or 2) sample size of 2500 with 80% random right censoring.

Maintenance Policy Optimization. Three preventive maintenance policies discussed in the literature were considered. The traditional approach of optimization through minimizing the cost functions requires the cost for a system failure (c_b) be explicitly known. However, c_b is often subjective and difficult to express in monetary terms. A "reliability cost model" was developed whereby system reliability for each policy was expressed as a function of cost. This technique allows an informed decision on the "optimal" policy based on a subjective assessment of c_b. Theoretical results were presented and the performance of the model applied to empirical data was assessed via a Monte Carlo simulation. Finally, the sensitivities of the respective policy costs with changes in the cost parameters were examined. This analysis revealed the nature of the relative policy costs as summarized below:

(1) The most cost efficient policy is irrespective of the desired reliability goal.

(2) The cost ratio $c_s/\sum c_i = 0.75$ represents the boundary between the ARP and BRP costs. The ratio $c_s/\sum c_i > 0.75$ favors the ARP, while $c_s/\sum c_i < 0.75$ favors the BRP.

(3) The ORP is always at least as cost efficient as either the ARP or the BRP. The ORP is degenerate with the ARP for large values of $c_s/\sum c_i$ and is degenerate with the BRP when $c_s/\sum c_i$ decreases to zero. The ORP offers the greatest advantage in cost efficiency over the other policies when $c_s/\sum c_i = 0.75$.

Suggestions for Future Research

MRL Statistical Tests. The MRL statistical tests considered in this research effort were determined to be unreliable when the data is subject to heavy random right censoring ($\geq 75\%$), even when the sample size is extremely large ($n = 2500$). A statistical test that it is more robust under these conditions is required. New tests motivated by the notion of virtual age could be pursued. Recall that $0 \leq t_v(t) \leq \mu$ for all t for an item with a NBUE distribution and $t_v(t_2) \geq t_v(t_1)$ for all $t_2 \geq t_1$ if an item is characterized with a DMRL function.

AMRAAM Preventive Maintenance. The MRL analysis of the AMRAAM failure data did not support the aging criterion. However, there was great disparity between the non-parametric and parametric results due to the lack of data in the tail of the non-parametric survivor function. This subject should be revisited as more data becomes available allowing better approximations of the system MRL function .

The AMRAAM failure data available for this research did not contain component level information. It is well known that complex systems often exhibit exponential lifetime distributions even though one or more components may be aging. Component MRL functions should be developed as data becomes available. Preventive maintenance policies at the component level could then be pursued if warranted.

Appendix A: MRL Example Code

I. Preliminaries

1. Set parameters and constants

Sample size: $n \equiv 30$

Weibull parameters: $\beta \equiv 1.5 \quad \alpha \equiv 1$

Plot time parameters: $tmax \equiv 3.0 \quad tstart \equiv 0.001 \quad tstep \equiv .05$

Smoothing parameter: $h \equiv 0.1$

Non-parametric or "semi-parametric" results: $switch \equiv 0$

switch = 0, non-parametric

switch = 1, semi-parametric

2. Weibull MRL and survivor functions (Leemis 1995:88):

The incomplete Gamma Function (Leemis 1995:286):

$$I(\alpha, x) := \frac{1}{\Gamma(\alpha)} \cdot \int_0^x y^{\alpha - 1} \cdot \exp(-y)\, dy$$

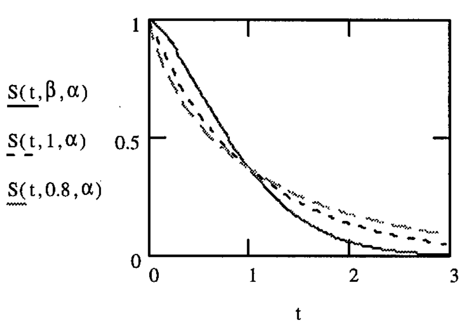
Survivor function:

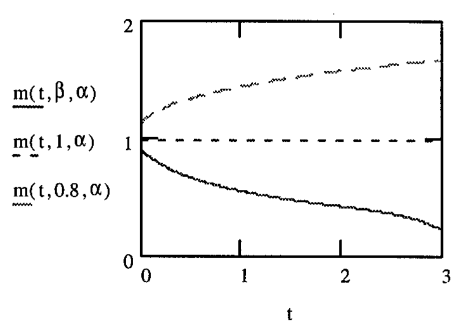
$$S(t, shape, scale) := \exp\left[-(scale \cdot t)^{shape}\right]$$

MRL function:

$$m(t, shape, scale) := \frac{\exp\left[(scale \cdot t)^{shape}\right]}{scale \cdot shape} \cdot \Gamma\left(\frac{1}{shape}\right) \cdot \left[1 - I\left[\frac{1}{shape}, (scale \cdot t)^{shape}\right]\right]$$

$t := 0.001, 0.02 .. tmax$





Weibull mean calculation: $\mu_{true} := \frac{1}{\beta \cdot \alpha} \cdot \Gamma\left(\frac{1}{\beta}\right) \qquad \mu_{true} = 0.903$

3. Weibull random variate generation:

$X := rweibull(n, \beta) \cdot \alpha \qquad U := sort(X) \qquad meanX := mean(X)$

```
Z := | i ← 1
     | while i ≤ n
     |   | Z_i ← U_(i-1)
     |   | i ← i + 1
     | Z
```

4. Weibull Parameter Estmation (Leemis, 1995:216):

a) Create set of ordered observed failure times:

```
oft(G) := | i ← 1
          | j ← 1
          | out ← G  if cols(G) = 1
          | while i ≤ n                if cols(G) = 2
          |   | if G_(i,1) = 1
          |   |   | out_(j,0) ← G_(i,0)
          |   |   | out_(j,1) ← i
          |   |   | j ← j + 1
          |   | i ← i + 1
          | out
```

b) shape parameter calculation:

estimation tolerance: $\varepsilon := 0.01$

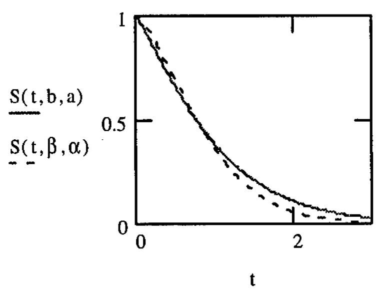
$$
\beta_n(G) := \left|
\begin{array}{l}
x \leftarrow \mathrm{oft}(G) \\
r \leftarrow \mathrm{rows}(x) - 1 \\
\beta_n \leftarrow \beta \\
g \leftarrow \left(\dfrac{r}{\beta_n} + \displaystyle\sum_{i=1}^{r} \ln\left(x_{i,0}\right) \right) - \dfrac{r \cdot \displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n} \cdot \ln\left(G_{i,0}\right)}{\displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n}} \\
g_{prime} \leftarrow -\dfrac{r}{\beta_n^{\,2}} - \dfrac{r \cdot \left[\left[\displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n} \right] \cdot \left[\displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n} \cdot \ln\left(G_{i,0}\right)^2 \right] - \left[\displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n} \cdot \ln\left(G_{i,0}\right) \right]^2 \right]}{\left[\displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n} \right]^2} \\
\beta_{new} \leftarrow \beta_n - \dfrac{g}{g_{prime}} \\
\text{while } \left|\beta_{new} - \beta_n\right| > \varepsilon \\
\quad \left|
\begin{array}{l}
\beta_n \leftarrow \beta_{new} \\
g \leftarrow \left(\dfrac{r}{\beta_n} + \displaystyle\sum_{i=1}^{r} \ln\left(x_{i,0}\right) \right) - \dfrac{r \cdot \displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n} \cdot \ln\left(G_{i,0}\right)}{\displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n}} \\
g_{prime} \leftarrow -\dfrac{r}{\beta_n^{\,2}} - \dfrac{r \cdot \left[\left[\displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n} \right] \cdot \left[\displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n} \cdot \ln\left(G_{i,0}\right)^2 \right] - \left[\displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n} \cdot \ln\left(G_{i,0}\right) \right]^2 \right]}{\left[\displaystyle\sum_{i=1}^{n} \left(G_{i,0}\right)^{\beta_n} \right]^2} \\
\beta_{new} \leftarrow \beta_n - \dfrac{g}{g_{prime}}
\end{array}
\right. \\
\beta_n
\end{array}
\right.
$$

c) scale parameter calculation:

$\beta_n(Z) = 1.222 \quad b := \beta_n(Z)$

$$\alpha_n(G,\beta) := \left| \begin{array}{l} num \leftarrow n \ \text{ if } \ cols(G) = 1 \\ num \leftarrow \sum G^{\langle 1 \rangle} \ \text{ if } \ cols(G) = 2 \\ out \leftarrow \left[\dfrac{num}{\sum_{i=1}^{n} (G_{i,0})^{\beta}} \right]^{\frac{1}{\beta}} \end{array} \right.$$

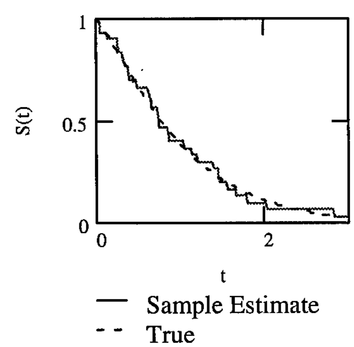
$\alpha_n(Z,\beta_n(Z)) = 0.953 \qquad a := \alpha_n(Z,\beta_n(Z))$



II. Complete Data Example

1. Sample Survivor Function (Leemis, 1995:253):

$$S_n(t) := \left| \begin{array}{l} i \leftarrow 1 \\ \left. \begin{array}{l} \text{while } i \le n \\ \quad \left| \begin{array}{l} \text{if } t \le Z_i \\ \quad \left| \begin{array}{l} s \leftarrow \dfrac{n-(i-1)}{n} \\ i \leftarrow n+1 \end{array} \right. \\ i \leftarrow i+1 \end{array} \right. \end{array} \right. \quad \text{if } t < Z_n \\ s \leftarrow 0 \ \text{ if } \ t > Z_n \\ s \end{array} \right.$$



2. Compute the semi-parametric MRL numerator constant:

$$C := \left| \begin{array}{l} C \leftarrow 0 \ \text{ if } \ switch = 0 \\ C \leftarrow \displaystyle\int_{Z_n}^{10 \cdot Z_n} S(u,\beta,\alpha)\, du \ \text{ if } \ switch = 1 \\ C \end{array} \right.$$

3. Sample MRL Function (Guess and Proschan, 1985:8-9):

$$
m_n(t,G) := \left|
\begin{array}{l}
j \leftarrow 1 \\
\text{while } j \le n \\
\quad \left|
\begin{array}{l}
\text{if } t < G_j \\
\quad \left|
\begin{array}{l}
k \leftarrow j - 1 \\
mrl \leftarrow \dfrac{\left[\displaystyle\sum_{i=k+1}^{n} \left(G_i - t\right) \right] + C}{n - k} \\
j \leftarrow n + 1
\end{array}
\right. \\
j \leftarrow j + 1
\end{array}
\right. \\
mrl
\end{array}
\right.
$$

4. Smooth Empirical MRL Function (Kalasekera, 1991):

Integrated Epanechnikov kernal function:

$$
W_k(u) := \left|
\begin{array}{l}
0.3354102u - 0.0223607u^3 + .5 \text{ if } |u| \le \sqrt{5} \\
1 \text{ if } u > \sqrt{5} \\
0 \text{ if } u < -\sqrt{5}
\end{array}
\right.
$$

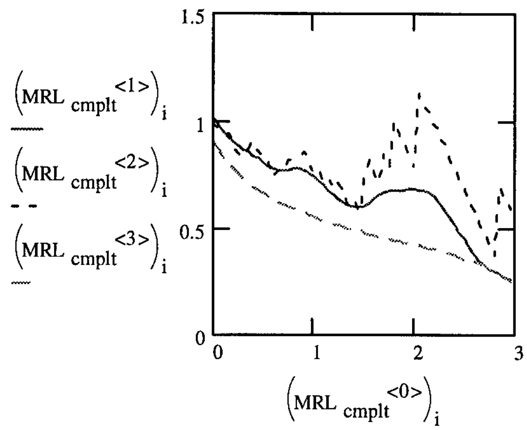
smoothed MRL function:

$$
m_{nk}(\tau,G,h) := \left|
\begin{array}{l}
out \leftarrow 0 \text{ if } \tau \ge G_n \\
out \leftarrow \dfrac{\left[meanX - \displaystyle\int_0^{\tau} 1 - \frac{1}{n} \cdot \sum_{j=1}^{n} W_k\left(\frac{x - G_j}{h}\right) dx \right] + C}{1 - \dfrac{1}{n+1} \cdot \left(\displaystyle\sum_{i=1}^{n} W_k\left(\frac{\tau - G_i}{h}\right) \right)} \text{ if } \tau < G_n \\
out
\end{array}
\right.
$$

5. Compute MRL Plot array

$$i := 0..\,\mathrm{rows}(MRL_{cmplt}) - 1$$

$$MRL_{cmplt} := \left|\begin{array}{l} t \leftarrow tstart \\ i \leftarrow 0 \\ \text{while } t \le tmax \\ \quad \left|\begin{array}{l} out_{i,0} \leftarrow t \\ out_{i,1} \leftarrow m_{nk}(t,Z,h) \\ out_{i,2} \leftarrow m_n(t,Z) \\ out_{i,3} \leftarrow m(t,\beta,\alpha) \\ t \leftarrow t + tstep \\ i \leftarrow i+1 \end{array}\right. \\ out \end{array}\right.$$



6. Hollander and Proschan (1975) V* Test:

$$c(k) := \left[\left[\left(\frac{4\cdot k^3}{3} - 4\cdot n\cdot k^2\right) + 3\cdot n^2\cdot k - \frac{n^3}{2}\right] + \frac{n^2}{2} - \frac{k^2}{2}\right] + \frac{k}{6}$$

$$V_{star} := \frac{\frac{1}{n^4}\cdot\sum_{i=1}^{n} c(i)\cdot Z_i}{meanX}$$

$$test := (210\cdot n)^{0.5}\cdot V_{star} \qquad pvalue := 1 - \mathrm{pnorm}(test, 0, 1)$$

$$test = 0.807 \qquad pvalue = 0.21$$

7. Aly (1990) Tn Test:

$$\sigma sqr_n := \frac{1}{n}\cdot\sum_{j=1}^{n}\left(1 + \ln\left(1 - \frac{j-1}{n}\right)\right)^2 \qquad \lambda_n := 1 + \frac{1}{n}\cdot\sum_{j=1}^{n}\ln\left(1 - \frac{j-1}{n}\right)$$

$$t_n := \frac{1}{meanX}\cdot\left[\sum_{i=1}^{n}\left[\left(1 + \ln\left(1 - \frac{i-1}{n}\right)\right)\cdot\left(1 - \frac{i-1}{n}\right)\cdot\left(Z_i - Z_{i-1}\right)\right]\right]$$

$$T_n := \frac{n^{\frac{1}{2}} \cdot \left(t_n - \lambda_n \cdot meanX\right)}{\sigma sqr_n^{\frac{1}{2}} \cdot meanX} \qquad T_n = 0.839 \quad pvalue := 1 - pnorm\left(T_n, 0, 1\right) \quad pvalue = 0.201$$

8. Ahmad (1992) Un Test

$$Upval(G) := \left| \begin{array}{l} U_n \leftarrow 0 \\ \text{for } j \in 2..n \\ \quad U_n \leftarrow \dfrac{\left[\displaystyle\sum_{i=1}^{j} \left(3 \cdot G_i - G_j\right)\right]}{\left(\dfrac{n!}{(n-2)!}\right)} + U_n \\ test \leftarrow (3 \cdot n)^{\frac{1}{2}} \cdot U_n \\ pval \leftarrow 1 - pnorm(|test|, 0, 1) \\ pval \end{array} \right. \qquad Upval(Z) = 0.037$$

III. Censored data Example

1. Generation of Random Censored Data:

exponential censoring distribution: $Y1 := rexp(n, 0.8)$

censored data pairs function:

$$rndcensor(G1, G2) := \left| \begin{array}{l} i \leftarrow 0 \\ \text{while } i \le n - 1 \\ \quad \left| \begin{array}{l} \text{if } G1_i \le G2_i \\ \quad \left| \begin{array}{l} U_{i,0} \leftarrow G1_i \\ U_{i,1} \leftarrow 1 \end{array} \right. \\ \text{if } G1_i > G2_i \\ \quad \left| \begin{array}{l} U_{i,0} \leftarrow G2_i \\ U_{i,1} \leftarrow 0 \end{array} \right. \\ i \leftarrow i + 1 \end{array} \right. \\ U \end{array} \right.$$

function to add a leading zero to the data set:

$$\text{add0}(G) := \left| \begin{array}{l} i \leftarrow 1 \\ \text{while } i \le n \\ \quad \left| \begin{array}{l} Z_{i,0} \leftarrow G_{i-1,0} \\ Z_{i,1} \leftarrow G_{i-1,1} \\ i \leftarrow i+1 \end{array} \right. \\ Z \end{array} \right.$$

the ordered random censored data set: $Zc := \text{add0}(\text{csort}(\text{rndcensor}(X, Y1), 0))$

compute censor percentage: $\text{percentcensor} := \dfrac{n - \sum Zc^{<1>}}{n}$ $\qquad \text{percentcensor} = 0.467$

last observation is considered "observed:" $\qquad Zc_{n,1} := 1$

2. KME Survivor Function

$$S_{KME}(G,t) := \left| \begin{array}{l} s \leftarrow 1 \text{ if } t \le G_{1,0} \\ \text{if } G_{1,0} < t \le G_{n,0} \\ \quad \left| \begin{array}{l} i \leftarrow 2 \\ \text{while } i \le n \\ \quad \left| \begin{array}{l} \text{if } t \le G_{i,0} \\ \quad \left| \begin{array}{l} k \leftarrow i \\ s \leftarrow 1 \\ \text{for } j \in 1, 2..k-1 \\ \quad s \leftarrow s \cdot \dfrac{n-j}{n-j+1} \text{ if } G_{j,1} = 1 \\ i \leftarrow n+1 \end{array} \right. \\ s \leftarrow 0 \text{ if } t > G_{n,0} \\ i \leftarrow i+1 \end{array} \right. \end{array} \right. \\ s \end{array} \right.$$

3. Smoothed KME Survivor Function (Kulasekera, 1990):

step size at each observation of SKME:

$$\text{step}(G,i) := \left| \begin{array}{l} S_{KME}\left(G, G_{i,0}\right) - S_{KME}\left(G, G_{i+1,0}\right) \text{ if } i < n \\ S_{KME}\left(G, G_{i,0}\right) \text{ if } i = n \end{array} \right.$$

the smoothed survivor function:

$$S_{sKME}(G,t) := \left\| \begin{array}{l} sum \leftarrow 0 \\ \text{for } j \in 1..n \\ \quad sum \leftarrow sum + W_k\left(\dfrac{t - G_{j,0}}{h}\right)\cdot step(G,j) \ \text{ if } G_{j,1} = 1 \\ S \leftarrow 1 - sum \\ S \end{array} \right.$$

4. PEXE Survivor Function (Kim and Proschan, 1991):

$$oft(G) := \left\| \begin{array}{l} i \leftarrow 1 \\ j \leftarrow 1 \\ \text{while } i \le n \\ \quad \left\| \begin{array}{l} \text{if } G_{i,1} = 1 \\ \quad \left\| \begin{array}{l} out_{j,0} \leftarrow G_{i,0} \\ out_{j,1} \leftarrow i \\ j \leftarrow j + 1 \end{array} \right. \\ i \leftarrow i + 1 \end{array} \right. \\ out \end{array} \right.$$

compute sample hazard function:

$$r_n(G,k) := \frac{1}{\displaystyle\sum_{i = oft(G)_{k-1,1}}^{oft(G)_{k,1} - 1} (n - i)\cdot\left(G_{i+1,0} - G_{i,0}\right)}$$

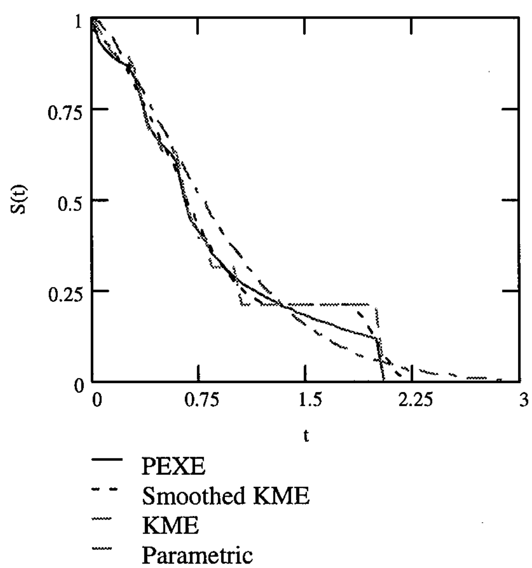
compute PEXE of S(t):

$$S_{PEXE}(G,t) := \left| \begin{array}{l} i \leftarrow 2 \\ s \leftarrow \exp\left(\left(-r_n(G,1)\cdot t\right)\right) \ \text{ if } \ t \le oft(G)_{1,0} \\ Z_{oft} \leftarrow oft(G) \\ \text{if } \ Z_{oft_{1,0}} < t \le G_{n,0} \\ \quad \left| \begin{array}{l} \text{while } \ i \le n \\ \quad \left| \begin{array}{l} \text{if } \ t \le Z_{oft_{i,0}} \\ \quad \left| \begin{array}{l} lamdat \leftarrow \displaystyle\sum_{k=1}^{i-1} r_n(G,k)\cdot\left(Z_{oft_{k,0}} - Z_{oft_{k-1,0}}\right) + \left(t - Z_{oft_{i-1,0}}\right)\cdot r_n(G,i) \\ i \leftarrow n+1 \end{array} \right. \\ i \leftarrow i+1 \end{array} \right. \\ s \leftarrow \exp((-lamdat)) \end{array} \right. \\ s \leftarrow 0 \ \text{ if } \ t > G_{n,0} \\ s \end{array} \right.$$

5. Compute Survivor Curve data array

$i := 0 .. rows(Srvr) - 1$

$$Srvr := \left| \begin{array}{l} t \leftarrow tstart \\ i \leftarrow 0 \\ \text{while } \ t \le tmax \\ \quad \left| \begin{array}{l} out_{i,0} \leftarrow t \\ out_{i,1} \leftarrow S_{PEXE}(Zc,t) \\ out_{i,2} \leftarrow S_{sKME}(Zc,t) \\ out_{i,3} \leftarrow S_{KME}(Zc,t) \\ out_{i,4} \leftarrow S(t,\beta,\alpha) \\ t \leftarrow t + tstep \\ i \leftarrow i+1 \end{array} \right. \\ out \end{array} \right.$$



6. Compute Semi-parametric MRL Function Numerator constant

compute Weibull parameter estimates: $a := \alpha_n\left(Zc, \beta_n(Zc)\right)$ $\quad a = 1.075$

$b := \beta_n(Zc)$ $\quad b = 1.315$

$$C := \left\| \begin{array}{l} C \leftarrow 0 \ \text{ if } \ switch = 0 \\ C \leftarrow \displaystyle\int_{Zc_{n,0}}^{10 \cdot Zc_{n,0}} S(u, b, a)\, du \ \text{ if } \ switch = 1 \\ C \end{array} \right.$$

7. KME MRL Function (Chen et. all, 1983):

$$
m_{KME}(G,t,C) := \left|\begin{array}{l}
i \leftarrow 1 \\
\text{while } i \le n-1 \\
\quad \left|\begin{array}{l}
\text{if } t \le G_{i,0} \\
\quad \left|\begin{array}{l}
S \leftarrow S_{KME}\left(G, G_{i,0}\right) \\
denom \leftarrow S \\
int \leftarrow S \cdot \left(G_{i,0} - t\right) \\
\text{for } j \in i+1, i+2 .. n \\
\quad \left|\begin{array}{l}
S \leftarrow S_{KME}\left(G, G_{j,0}\right) \ \text{ if } G_{j-1,1} = 1 \\
int \leftarrow int + S \cdot \left(G_{j,0} - G_{j-1,0}\right)
\end{array}\right. \\
mrl \leftarrow \dfrac{int + C}{denom} \\
i \leftarrow n+1
\end{array}\right. \\
i \leftarrow i+1
\end{array}\right. \\
mrl \leftarrow \dfrac{S_{KME}\left(G, G_{n,0}\right) \cdot \left(G_{n,0} - t\right) + C}{S_{KME}(G,t)} \ \text{ if } G_{n-1,0} < t \le G_{n,0} \\
mrl
\end{array}\right.
$$

8. Smooth KME MRL Function (Kulasekera, 1990):

$$
\mu_{sKME}(G) := \int_0^{G_{n,0}} S_{sKME}(G,u)\, du \qquad\qquad \mu := \mu_{sKME}(Zc)
$$

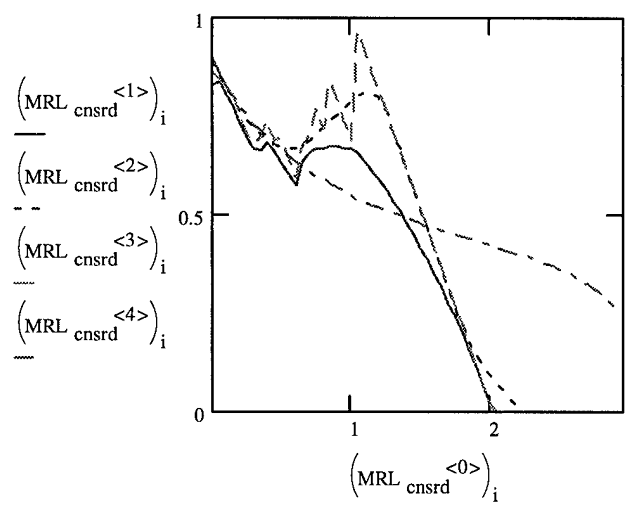
$$
m_{sKME}(G,t,\mu,C) := \left|\begin{array}{l}
S \leftarrow S_{sKME}(G,t) \\
\text{if } switch = 0 \\
\quad \left|\begin{array}{l}
out \leftarrow 0 \ \text{ if } S \le 0 \\
out \leftarrow \dfrac{\left(\mu - \displaystyle\int_0^t S_{sKME}(G,u)\, du\right) + C}{S} \ \text{ otherwise}
\end{array}\right. \\
\text{if } switch = 1 \\
\quad \left|\begin{array}{l}
out \leftarrow 0 \ \text{ if } t \ge G_{n,0} \\
out \leftarrow \dfrac{\left(\mu - \displaystyle\int_0^t S_{sKME}(G,u)\, du\right) + C}{S} \ \text{ otherwise}
\end{array}\right. \\
out
\end{array}\right.
$$

9. PEXE MRL estimate (Joe and Proschan, 1981):

$$m_{PEXE}(G,t,C) := \left| \begin{array}{l} mrl \leftarrow \dfrac{\displaystyle\int_{t}^{G_{n,0}} S_{PEXE}(G,y)\,dy + C}{S_{PEXE}(G,t)} \quad \text{if } t<G_{n,0} \\ mrl \leftarrow 0 \quad \text{otherwise} \\ mrl \end{array} \right.$$

10. Compute MRL Plot Data Array

$$MRL_{cnsrd} := \left| \begin{array}{l} t \leftarrow tstart \\ i \leftarrow 0 \\ \text{while } t \le tmax \\ \quad \left| \begin{array}{l} out_{i,0} \leftarrow t \\ out_{i,1} \leftarrow m_{PEXE}(Zc,t,C) \\ out_{i,2} \leftarrow m_{sKME}(Zc,t,\mu,C) \\ out_{i,3} \leftarrow m_{KME}(Zc,t,C) \\ out_{i,4} \leftarrow m(t,\beta,\alpha) \\ t \leftarrow t + tstep \\ i \leftarrow i+1 \end{array} \right. \\ out \end{array} \right.$$



11. Chen et. all (1983) DMRL test with censored data

$$c(u) := \frac{n-u}{n-u+1}$$

$$\Delta F_n := \sum_{i=1}^{n} \left[-\frac{1}{6} \cdot \prod_{j=1}^{i-1} c(j)^{Zc_{j,1}} + \left(\frac{1}{2} \cdot \prod_{j=1}^{i-1} c(j)^{2 \cdot Zc_{j,1}} - \frac{1}{3} \cdot \prod_{j=1}^{i-1} c(j)^{4 \cdot Zc_{j,1}} \right) \right] \cdot \left(Zc_{i,0} - Zc_{i-1,0} \right)$$

$$\mu_{KME}(G) := \begin{array}{|l} S \leftarrow 1 \\ int \leftarrow G_{1,0} \\ \text{for } i \in 2..n \\ \quad \begin{array}{|l} S \leftarrow S_{KME}(G, G_{i,0}) \text{ if } G_{i-1,1} = 1 \\ int \leftarrow int + S \cdot \left(G_{i,0} - G_{i-1,0} \right) \end{array} \\ int \end{array}$$

$$\mu hat_n := \mu_{KME}(Zc)$$

$$Vc := \frac{\Delta F_n}{\mu hat_n} \qquad B(i,a) := \exp\left(-\frac{a}{\mu hat_n} \cdot Zc_{i,0} \right)$$

$$half := n \cdot \left(\frac{B(n,2)}{72} - \frac{B(n,3)}{18} + \frac{B(n,4)}{16} + \frac{B(n,5)}{45} - \frac{B(n,6)}{18} + \frac{B(n,8)}{72} \right)$$

$$\tau sqr := \frac{1}{720} + \sum_{i=1}^{n-1} \frac{n \cdot \left(\frac{B(i,2)}{72} - \frac{B(i,3)}{18} + \frac{B(i,4)}{16} + \frac{B(i,5)}{45} - \frac{B(i,6)}{18} + \frac{B(i,8)}{72} \right)}{(n-i+1) \cdot (n-i)} + half$$

$$test := \frac{n^{\frac{1}{2}} \cdot Vc}{\tau sqr^{\frac{1}{2}}} \qquad pvalue := 1 - pnorm(test, 0, 1) \qquad pvalue = 0.24 \quad test = 0.708$$

12. Lim and Koh (1996) NBUE Test :

$$Lc := \frac{1}{\mu hat_n} \cdot \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} c(j)^{Zc_{j,1}} \right) \cdot \left(\ln\left(\prod_{j=1}^{i-1} c(j)^{Zc_{j,1}} + 1 \right) \right) \cdot \left(Zc_{i,0} - Zc_{i-1,0} \right)$$

$$half := n \cdot \left[\left(\frac{1}{4} - \frac{Zc_{n,0}}{2 \cdot \mu hat_n} \right) + \frac{\left(Zc_{n,0} \right)^2}{2 \cdot \mu hat_n^{\,2}} \right] \cdot B(n,4)$$

$$\sigma sqr := \frac{1}{4} + \sum_{i=1}^{n-1} \frac{n}{(n-i+1) \cdot (n-i)} \cdot \left[\left(\frac{1}{4} - \frac{Zc_{i,0}}{2 \cdot \mu hat_n} \right) + \frac{\left(Zc_{i,0} \right)^2}{2 \cdot \mu hat_n^{\,2}} \right] \cdot B(i,4) - half$$

$$test := \frac{n^{\frac{1}{2}} \cdot Lc}{\sigma sqr^{\frac{1}{2}}} \qquad pvalue := 1 - pnorm(test, 0, 1) \qquad test = 3.981 \quad pvalue = 3.433 \cdot 10^{-5}$$

13. Lim and Park (1993) NBUE Test :

$$\delta_n := \sum_{i=1}^{n} \left(2 \cdot \prod_{j=1}^{i-1} c(j)^{Zc_{j,1} \cdot 2} - \prod_{j=1}^{i-1} c(j)^{Zc_{j,1}} \right) \cdot \left(Zc_{i,0} - Zc_{i-1,0} \right)$$

$$\delta pval\left(G, \delta_n, \mu\right) := \left| \begin{array}{l} half1 \leftarrow \frac{1}{6} + \sum_{i=1}^{n-1} \frac{n}{(n-i+1)\cdot(n-i)} \cdot \left[\left(B(i,4,G,\mu) - \frac{4 \cdot B(i,3,G,\mu)}{3} \right) + \frac{B(i,2,G,\mu)}{2} \right] \\ half2 \leftarrow n \cdot \left[\left(B(n,4,G,\mu) - \frac{4}{3} \cdot B(n,3,G,\mu) \right) + \frac{1}{2} \cdot B(n,2,G,\mu) \right] \\ \tau sqr \leftarrow half1 - half2 \\ test \leftarrow \frac{n^{\frac{1}{2}} \cdot \frac{\delta_n}{\mu}}{\tau sqr^{\frac{1}{2}}} \\ pval \leftarrow 1 - pnorm(test, 0, 1) \\ pval \end{array} \right.$$

$$\delta pval\left(Zc, \delta_n, \mu hat_n\right) = 0.074$$

Appendix B: Statistaical Test Comparison Code

Complete Data Tests

Tn (Aly), Un (Ahmad), and V* (Hollandar and Proschan) Tests for D(I)MRL/NBUE with Complete Data

1. Set Constants:

$n \equiv 100$	sample size
$\beta min \equiv 0.9$	weibull shape param min
$\beta max \equiv 1.5$	weibull shape param max
$delta\beta \equiv 0.1$	weibull shape param step size
$\alpha \equiv 1.0$	weibull scale param
$nreps \equiv 30$	num of repititions at each β

2. Hollander and Proschan (1975) V* Test subroutine:

$$c(k) := \left[\left[\left(\frac{4 \cdot k^3}{3} - 4 \cdot n \cdot k^2\right) + 3 \cdot n^2 \cdot k - \frac{n^3}{2}\right] + \frac{n^2}{2} - \frac{k^2}{2}\right] + \frac{k}{6}$$

$$\mathrm{Vpval}(G, mean) := \left| \begin{array}{l} V_{star} \leftarrow \dfrac{\dfrac{1}{n^4} \cdot \displaystyle\sum_{i=1}^{n} c(i) \cdot G_i}{mean} \\ test \leftarrow (210 \cdot n)^{0.5} \cdot V_{star} \\ pval \leftarrow 1 - \mathrm{pnorm}(test, 0, 1) \\ pval \end{array} \right.$$

3. Aly (1990) Tn Test subroutine:

$$
\mathrm{Tpval}(G,\mathrm{mean}) := \left|\begin{array}{l}
t_n \leftarrow \dfrac{1}{\mathrm{mean}} \cdot \left[\displaystyle\sum_{i=1}^{n} \left[\left(1+\ln\left(1-\frac{i-1}{n}\right)\right)\cdot\left(1-\frac{i-1}{n}\right)\cdot\left(G_i - G_{i-1}\right)\right]\right] \\
\lambda_n \leftarrow 1 + \dfrac{1}{n} \cdot \displaystyle\sum_{j=1}^{n} \ln\left(1-\frac{j-1}{n}\right) \\
\sigma sqr_n \leftarrow \dfrac{1}{n} \cdot \displaystyle\sum_{j=1}^{n} \left(1+\ln\left(1-\frac{j-1}{n}\right)\right)^2 \\
T_n \leftarrow \dfrac{n^{\frac{1}{2}} \cdot \left(t_n - \lambda_n \cdot \mathrm{mean}\right)}{\sigma sqr_n^{\frac{1}{2}} \cdot \mathrm{mean}} \\
\mathrm{pval} \leftarrow 1 - \mathrm{pnorm}\left(T_n, 0, 1\right) \\
\mathrm{pval}
\end{array}\right.
$$

4. Ahmad Un Test Subroutine (1992)

$$
\mathrm{Upval}(G) := \left|\begin{array}{l}
U_n \leftarrow 0 \\
\text{for } j \in 2..n \\
\quad U_n \leftarrow \dfrac{\left[\displaystyle\sum_{i=1}^{j} \left(3 \cdot G_i - G_j\right)\right]}{n \cdot (n-1)} + U_n \\
\mathrm{test} \leftarrow (3 \cdot n)^{\frac{1}{2}} \cdot U_n \\
\mathrm{pval} \leftarrow 1 - \mathrm{pnorm}(\mathrm{test}, 0, 1) \\
\mathrm{pval}
\end{array}\right.
$$

5. Add an initial zero to the ordered data set:

$$
\mathrm{addzero}(G) := \left|\begin{array}{l}
i \leftarrow 1 \\
\text{while } i \le n \\
\quad \left|\begin{array}{l} Z_i \leftarrow G_{i-1} \\ i \leftarrow i + 1 \end{array}\right. \\
Z
\end{array}\right.
$$

6. Compute p-value comparison matrix

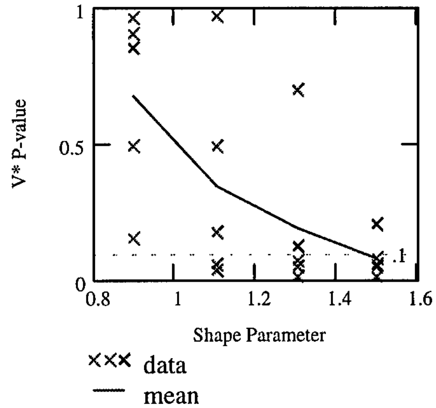
```
testcompare := | β ← βmin
               | i ← 0
               | βcount ← 0
               | while β ≤ βmax + deltaβ/2
               |   | while i ≤ βcount + nreps − 1
               |   |   | data ← rweibull(n, β)·α
               |   |   | meanX ← mean(data)
               |   |   | Z ← addzero(sort(data))
               |   |   | out[i,0 ← β
               |   |   | out[i,1 ← Vpval(Z, meanX)
               |   |   | out[i,2 ← Tpval(Z, meanX)
               |   |   | out[i,3 ← Upval(Z)
               |   |   | i ← i + 1
               |   | βcount ← i
               |   | β ← β + deltaβ
               | out
```

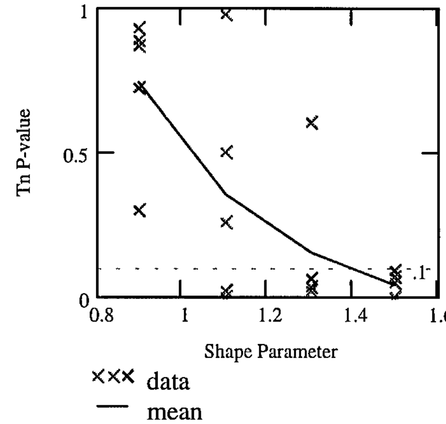
7. Plot the results:

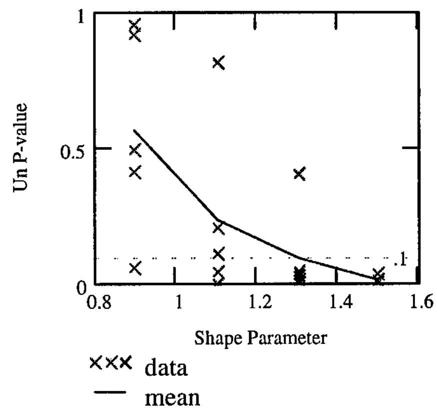
$i := 0..\mathrm{rows}(\mathrm{testcompare}) - 1$

```
avg := | imax ← (βmax − βmin)/deltaβ
       | for i ∈ 0, 1 .. imax
       |   | jmin ← nreps·i
       |   | jmax ← jmin + nreps − 1
       |   | out[i,0 ← mean(submatrix(testcompare, jmin, jmax, 0, 0))
       |   | out[i,1 ← mean(submatrix(testcompare, jmin, jmax, 1, 1))
       |   | out[i,2 ← mean(submatrix(testcompare, jmin, jmax, 2, 2))
       |   | out[i,3 ← mean(submatrix(testcompare, jmin, jmax, 3, 3))
       | out
```

$j := 0..\mathrm{rows}(\mathrm{avg}) - 1$







Censored Data Tests

Tn (Lim and Park), Vc (Chen, Hollandar and Langberg), and Lc (Lim and Koh) tests for NBUE / DMRL with Censored Data

1. Set Constants:

$n \equiv 30$	sample size	$\beta min \equiv 0.9$	weibull shape param min
$\lambda \equiv 0.5$	exp censor rate	$\beta max \equiv 1.5$	weibull shape param max
$\alpha \equiv 1.0$	weibull scale param	$delta\beta \equiv 0.2$	weibull shape param step size
$\lambda \equiv 0.5$	exp censor rate		
$nreps \equiv 5$	num of repititions at each β		

2. Chen et. all (1983) DMRL test with censored data :

$$c(u) := \frac{n-u}{n-u+1} \qquad B(i, a, G, mean) := \exp\left(-\frac{a}{mean} \cdot G_{i,0}\right)$$

$$\mu_{KME}(G) := \left| \begin{array}{l} S \leftarrow 1 \\ int \leftarrow G_{1,0} \\ \text{for } i \in 2..n \\ \quad \left| \begin{array}{l} S \leftarrow S_{KME}\left(G, G_{i,0}\right) \text{ if } G_{i-1,1} = 1 \\ int \leftarrow int + S \cdot \left(G_{i,0} - G_{i-1,0}\right) \end{array} \right. \\ int \end{array} \right.$$

$$Vpval(G, Vc, \mu) := \left| \begin{array}{l} half1 \leftarrow n \cdot \left(\frac{B(n,2,G,\mu)}{72} - \frac{B(n,3,G,\mu)}{18} + \frac{B(n,4,G,\mu)}{16} + \frac{B(n,5,G,\mu)}{45} - \frac{B(n,6,G,\mu)}{18} + \frac{B(n,8,G,\mu)}{72} \right) \\ half2 \leftarrow \sum_{i=1}^{n-1} \frac{n \cdot \left(\frac{B(i,2,G,\mu)}{72} - \frac{B(i,3,G,\mu)}{18} + \frac{B(i,4,G,\mu)}{16} + \frac{B(i,5,G,\mu)}{45} - \frac{B(i,6,G,\mu)}{18} + \frac{B(i,8,G,\mu)}{72} \right)}{(n-i+1) \cdot (n-i)} \\ \tau sqr \leftarrow \frac{1}{720} + half1 + half2 \\ test \leftarrow \frac{n^{\frac{1}{2}} \cdot Vc}{\tau sqr^{\frac{1}{2}}} \\ pval \leftarrow 1 - pnorm(test, 0, 1) \\ pval \end{array} \right.$$

3. Lim and Park (1993) NBUE Test with randomly censored data :

$$\delta pval\left(G, \delta_n, \mu\right) := \left| \begin{array}{l} half1 \leftarrow \frac{1}{6} + \sum_{i=1}^{n-1} \frac{n}{(n-i+1) \cdot (n-i)} \cdot \left[\left(B(i,4,G,\mu) - \frac{4 \cdot B(i,3,G,\mu)}{3} \right) + \frac{B(i,2,G,\mu)}{2} \right] \\ half2 \leftarrow n \cdot \left[\left(B(n,4,G,\mu) - \frac{4}{3} \cdot B(n,3,G,\mu) \right) + \frac{1}{2} \cdot B(n,2,G,\mu) \right] \\ \tau sqr \leftarrow half1 - half2 \\ test \leftarrow \frac{n^{\frac{1}{2}} \cdot \delta_n}{\tau sqr^{\frac{1}{2}}} \\ pval \leftarrow 1 - pnorm(test, 0, 1) \\ pval \end{array} \right.$$

4. Lim and Koh (1996) NBUE Test with randomly censored data:

$$
\mathrm{Lpval}(G,\mu) := \left|
\begin{array}{l}
Lc \leftarrow \dfrac{1}{\mu}\cdot\left[\displaystyle\sum_{i=1}^{n}\left[\left(\prod_{j=1}^{i-1} c(j)^{G_{j,1}}\right)\cdot\left(\ln\left(\prod_{j=1}^{i-1} c(j)^{G_{j,1}}+1\right)\right)\cdot\left(G_{i,0}-G_{i-1,0}\right)\right]\right] \\
half \leftarrow n\cdot\left[\left(\dfrac{1}{4}-\dfrac{G_{n,0}}{2\cdot\mu}\right)+\dfrac{\left(G_{n,0}\right)^2}{2\cdot\mu^2}\right]\cdot B(n,4,G,\mu) \\
\sigma sqr \leftarrow \dfrac{1}{4}+\left[\displaystyle\sum_{i=1}^{n-1}\left[\dfrac{n}{(n-i+1)\cdot(n-i)}\cdot\left[\left(\dfrac{1}{4}-\dfrac{G_{i,0}}{2\cdot\mu}\right)+\dfrac{\left(G_{i,0}\right)^2}{2\cdot\mu^2}\right]\cdot B(i,4,G,\mu)\right]\right]-half \\
test \leftarrow \dfrac{n^{\frac{1}{2}}\cdot Lc}{\sigma sqr^{\frac{1}{2}}} \\
pval \leftarrow 1-\mathrm{pnorm}(test,0,1) \\
pval
\end{array}\right.
$$

5. Create Censored Data Pairs :

$$
\mathrm{rndcensor}(G1,G2) := \left|
\begin{array}{l}
i \leftarrow 0 \\
\text{while } i \le n-1 \\
\quad \left| \begin{array}{l}
\text{if } G1_i \le G2_i \\
\quad \left| \begin{array}{l} U_{i,0} \leftarrow G1_i \\ U_{i,1} \leftarrow 1 \end{array}\right. \\
\text{if } G1_i > G2_i \\
\quad \left| \begin{array}{l} U_{i,0} \leftarrow G2_i \\ U_{i,1} \leftarrow 0 \end{array}\right. \\
i \leftarrow i+1
\end{array}\right. \\
U
\end{array}\right.
$$

$$
\mathrm{addzero}(G) := \left|
\begin{array}{l}
i \leftarrow 1 \\
\text{while } i \le n \\
\quad \left| \begin{array}{l}
Z_{i,0} \leftarrow G_{i-1,0} \\
Z_{i,1} \leftarrow G_{i-1,1} \\
i \leftarrow i+1
\end{array}\right. \\
Z
\end{array}\right.
$$

$$
\mathrm{data}(shape) := \left|
\begin{array}{l}
X \leftarrow \mathrm{rweibull}(n,shape)\cdot\alpha \\
Y \leftarrow \mathrm{rexp}(n,\lambda) \\
temp \leftarrow \mathrm{rndcensor}(X,Y) \\
Zctemp \leftarrow \mathrm{csort}(temp,0) \\
Zc \leftarrow \mathrm{addzero}(Zctemp) \\
Zc_{n,1} \leftarrow 1 \\
Zc
\end{array}\right.
$$

6. Compute Censor Proportion:

$$\text{cnsrproportion}(G) := \left| \begin{array}{l} \text{num} \leftarrow \sum G^{\langle 1 \rangle} \\ \text{out} \leftarrow \dfrac{n - \text{num}}{n} \\ \text{out} \end{array} \right.$$

7. Compute Products for Test Stat Computation

```
prod(G, stop) := | out_0 ← 1
                 | out_1 ← 1
                 | out_2 ← 1
                 | if stop ≥ 1
                 |   | for j ∈ 1..stop
                 |   |   if G_{j,1} = 1
                 |   |     | c_j ← c(j)
                 |   |     | out_0 ← c_j · out_0
                 |   |     | out_1 ← c_j^2 · out_1
                 |   |     | out_2 ← c_j^4 · out_2
                 |   | dummy ← j
                 | out
```

8. Compute P-value Comparison Matrix

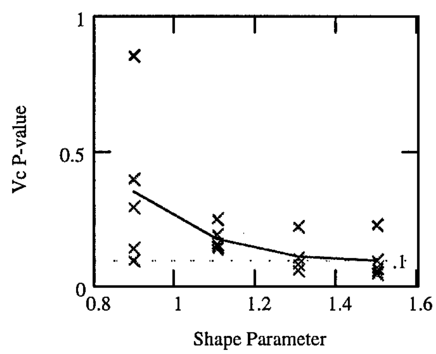
$$
\text{testcompare} := \left|
\begin{array}{l}
\beta\text{count} \leftarrow 0 \\
\beta \leftarrow \beta\text{min} \\
j \leftarrow 0 \\
\text{while } \beta \le \beta\text{max} + \dfrac{\text{delta}\beta}{2} \\
\quad \left|
\begin{array}{l}
\text{while } j \le \beta\text{count} + \text{nreps} - 1 \\
\quad \left|
\begin{array}{l}
Z \leftarrow \text{data}(\beta) \\
\mu \leftarrow \mu_{\text{KME}}(Z) \\
\Delta F \leftarrow 0 \\
\delta F_{n} \leftarrow 0 \\
\text{for } i \in 1..n \\
\quad \left|
\begin{array}{l}
\text{prd} \leftarrow \text{prod}(Z, i-1) \\
\Delta F \leftarrow \Delta F + \left[-\dfrac{1}{6}\cdot\text{prd}_0 + \left(\dfrac{1}{2}\cdot\text{prd}_1 - \dfrac{1}{3}\cdot\text{prd}_2\right)\right]\cdot\left(Z_{i,0} - Z_{i-1,0}\right) \\
\delta F_{n} \leftarrow \delta F_{n} + \left(2\cdot\text{prd}_1 - \text{prd}_0\right)\cdot\left(Z_{i,0} - Z_{i-1,0}\right)
\end{array}
\right. \\
\text{Vc} \leftarrow \dfrac{\Delta F}{\mu} \\
\delta_{n} \leftarrow \dfrac{\delta F_{n}}{\mu} \\
\text{out}_{j,0} \leftarrow \beta \\
\text{out}_{j,1} \leftarrow \text{Vpval}(Z, \text{Vc}, \mu) \\
\text{out}_{j,2} \leftarrow \delta\text{pval}\left(Z, \delta_{n}, \mu\right) \\
\text{out}_{j,3} \leftarrow \text{cnsrproportion}(Z) \\
j \leftarrow j + 1
\end{array}
\right. \\
\beta \leftarrow \beta + \text{delta}\beta \\
\beta\text{count} \leftarrow j
\end{array}
\right. \\
\text{out}
\end{array}
\right.
$$

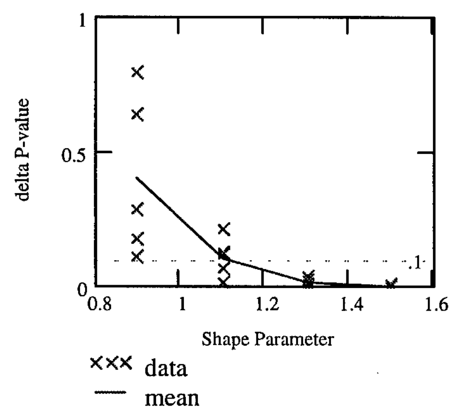
$\text{censorlvl} := \text{mean}\left(\text{testcompare}^{\langle 3\rangle}\right)$ $\qquad$ $\text{censorlvl} = 0.34$

9. Plot the Results:

$i := 0..\,\mathrm{rows(testcompare)} - 1$

$$
avg := \left|
\begin{array}{l}
imax \leftarrow \dfrac{\beta max - \beta min}{delta\beta} \\
\text{for } i \in 0, 1 .. imax \\
\quad \left|
\begin{array}{l}
jmin \leftarrow nreps \cdot i \\
jmax \leftarrow jmin + nreps - 1 \\
out_{i,0} \leftarrow \mathrm{mean(submatrix(testcompare}, jmin, jmax, 0, 0)) \\
out_{i,1} \leftarrow \mathrm{mean(submatrix(testcompare}, jmin, jmax, 1, 1)) \\
out_{i,2} \leftarrow \mathrm{mean(submatrix(testcompare}, jmin, jmax, 2, 2))
\end{array}
\right. \\
out
\end{array}
\right.
$$





Appendix C: Reliability Cost Model Code

Theoretical Reliability Cost Model

1. Component Distribution Input Parameters

Weibull scale params $\alpha := (0\ \ 0.0003034\ \ 0.0002716\ \ 0.0002848\ \ 0.0002736)^T$

Weibull shape params $\beta := (0\ \ 1.2\ \ 1.3\ \ 1.4\ \ 1.5)^T$

2. Cost Coefficients

component costs: $c := (0\ \ .3\ \ .25\ \ .2\ \ .15)^T$

setup repair cost: $c_s := .1$

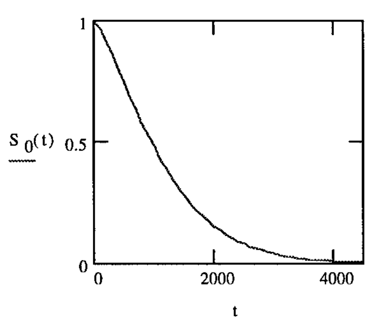
system replacement cost: $c_p := c_s + \sum c$ $\qquad c_p = 1$

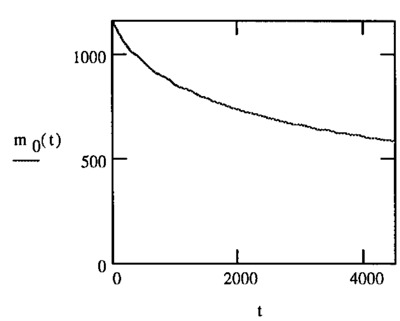
component replacement costs: $c_f := \begin{bmatrix} 0 \\ c_s + c_1 \\ c_s + c_2 \\ c_s + c_3 \\ c_s + c_4 \end{bmatrix} \qquad c_f = \begin{bmatrix} 0 \\ 0.4 \\ 0.35 \\ 0.3 \\ 0.25 \end{bmatrix}$

3. Weibull Survivor and MRL Functions (from MRL example worksheet)

4. System Survivor and MRL Functions $\quad tmax \equiv 4500 \quad t := 0.000, 100 .. tmax$

$$S_0(t) := \exp\left[-\sum_{i=1}^{4} \left(\alpha_i \cdot t\right)^{\beta_i}\right] \qquad inf := 15000 \qquad m_0(t) := \frac{1}{S_0(t)} \cdot \int_t^{inf} S_0(u)\, du$$





5. Function to determine the Reliability Goal given T

"Virtual" replacement period: $T_V(T) := m_0(0) - m_0(T)$

System reliability: $R(T) := S_0\left(T_V(T)\right)$

6. Normalized Age Replacement Policy Cost Model

$$C_A(T) := \frac{1}{\displaystyle\int_0^T S_0(u)\,du}$$

7. Component renewal function:

Weibull mean calculation: $\mu(\beta,\alpha) := \frac{1}{\beta\cdot\alpha}\cdot\Gamma\left(\frac{1}{\beta}\right)$

Weibull variance calculation: $var(\beta,\alpha) := \frac{1}{(\alpha)^2}\cdot\left[\frac{2}{\beta}\cdot\Gamma\left(\frac{2}{\beta}\right) - \left(\frac{1}{\beta}\cdot\Gamma\left(\frac{1}{\beta}\right)\right)^2\right]$

Weibull density function: $f(t,\beta,\alpha) := \beta\cdot(\alpha)^{\beta}\cdot t^{\beta-1}\cdot\exp\left[-(\alpha\cdot t)^{\beta}\right]$

Large t renewal function approximation:

$$W_1(t,\beta,\alpha) := \left\| \begin{array}{l} \mu \leftarrow \mu(\beta,\alpha) \\ \left(1 + \frac{t}{\mu} + \frac{var(\beta,\alpha) - \mu^2}{2\cdot\mu^2}\right)\cdot(1 - S(t,\beta,\alpha)) - \frac{1}{\mu}\cdot\displaystyle\int_0^t s\cdot f(s,\beta,\alpha)\,ds \end{array}\right.$$

Component renewal function: $W(t,\beta,\alpha) := \max\left(\left(1 - S(t,\beta,\alpha)\quad W_1(t,\beta,\alpha)\right)\right)$

8. Normalized Block Replacement Policy Cost Model

$$C_B(T) := \frac{1 + \displaystyle\sum_{i=1}^{4} \frac{c_{f_i}}{c_p}\cdot W\left(T,\beta_i,\alpha_i\right)}{T}$$

9. Normalized Opportunistic Replacement Policy Cost Model

```
C_O(T) := | τ ← 0
          | τ_step ← 100
          | out_0 ← C_A(T)
          | out_1 ← τ
          | while τ ≤ T
          |   | denom ← τ + ∫_0^(T−τ) exp[ [ Σ_(i=1)^4 −[(τ + u)·α_i]^(β_i) + (τ·α_i)^(β_i) ] ] du
          |   | new ← (1/denom)·( 1 + Σ_(i=1)^4 (c_(f_i)/c_p)·W(τ, β_i, α_i) )
          |   | if new < out_0
          |   |   | out_0 ← new
          |   |   | out_1 ← τ
          |   | τ ← τ + τ_step
          | out
```

$$\mathrm{denom} \leftarrow \tau + \int_0^{T-\tau} \exp\left[\left[\sum_{i=1}^{4} -\left[(\tau+u)\cdot\alpha_i\right]^{\beta_i} + \left(\tau\cdot\alpha_i\right)^{\beta_i}\right]\right] du$$

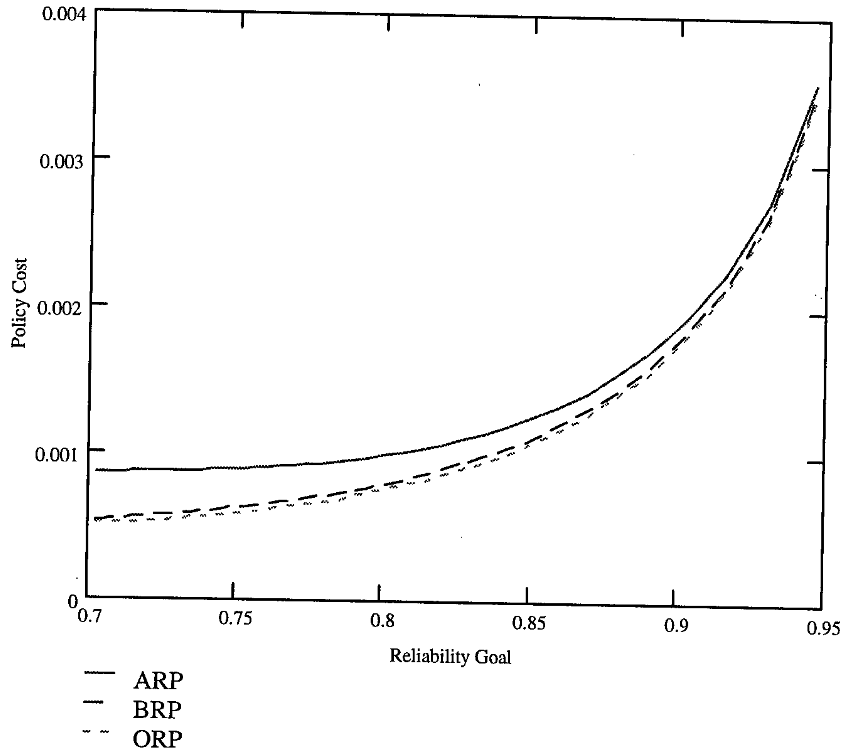
$$\mathrm{new} \leftarrow \frac{1}{\mathrm{denom}}\cdot\left(1 + \sum_{i=1}^{4} \frac{c_{f_i}}{c_p}\cdot W\left(\tau,\beta_i,\alpha_i\right)\right)$$

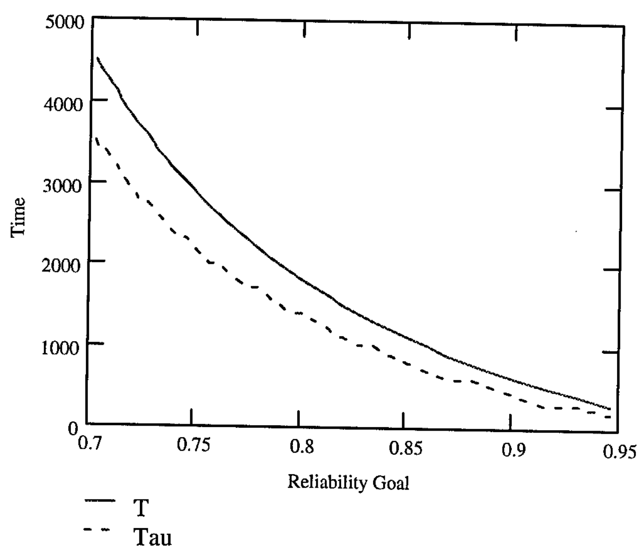
10. Policy Costs vs Reliability

Tmax := 4500 Tstep := 100

```
COST := | i ← 0
        | T ← Tmax
        | while T > 0.001
        |   | out_(i,0) ← T
        |   | out_(i,1) ← T_v(T)
        |   | out_(i,2) ← R(T)
        |   | out_(i,3) ← C_A(T)
        |   | out_(i,4) ← C_B(T)
        |   | out_(i,5) ← C_O(T)_0
        |   | out_(i,6) ← C_O(T)_1
        |   | T ← T − Tstep
        |   | i ← i + 1
        | out
```

i := 0.. rows(COST) − 3

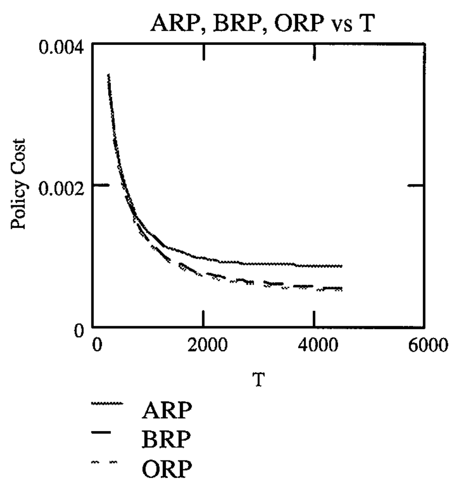


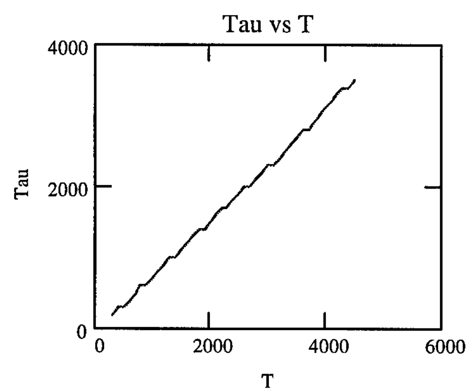


Cost Analysis

$Tmax := 4500$ $Tstep := 100$

$$COST := \left\| \begin{array}{l} i \leftarrow 0 \\ T \leftarrow Tmax \\ \text{while } T>0.001 \\ \quad \left\| \begin{array}{l} out_{i,0} \leftarrow T \\ out_{i,1} \leftarrow C_A(T) \\ out_{i,2} \leftarrow C_B(T) \\ out_{i,3} \leftarrow C_O(T)_0 \\ out_{i,4} \leftarrow C_O(T)_1 \\ T \leftarrow T - Tstep \\ i \leftarrow i+1 \end{array} \right. \\ out \end{array} \right.$$





Empirical Reliability Goal

1. Model Inputs.

sample size: $n \equiv 50$
Weibull param. estimation:
shape param initial guess: $\beta_0 \equiv 1.3$
shape param tolerance: $\varepsilon \equiv .02$

exponential censor rate: $\lambda \equiv .0018$
smoothed KME bandwidth: $h \equiv 100$
number of replications: $nreps \equiv 4$
replacement period: $Tmax := 1000$ $\quad Tstep := 250$ $\quad Tmin := 500$

2. DATA Set Generation.

```
DATA(α,β) := | X_1 ← rweibull(n,β_1)·(1/α_1)
             | X_2 ← rweibull(n,β_2)·(1/α_2)
             | X_3 ← rweibull(n,β_3)·(1/α_3)
             | X_4 ← rweibull(n,β_4)·(1/α_4)
             | X_c ← augment(X_1,X_2)
             | X_c ← augment(X_c,X_3)
             | X_c ← augment(X_c,X_4)
             | Y ← rexp(n,λ)
             | for i ∈ 1..n
             |   | X_{i,0} ← min((Xc_{i-1,0}  Xc_{i-1,1}  Xc_{i-1,2}  Xc_{i-1,3}  Y_{i-1}))
             |   | X_{i,1} ← 1  if X_{i,0} < Y_{i-1}
             |   | X_{i,1} ← 0  otherwise
             |   | for j ∈ 1..4
             |   |   | if Xc_{i-1,j-1} > X_{i,0}
             |   |   |   | X_{i,2·j} ← X_{i,0}
             |   |   |   | X_{i,2·j+1} ← 0
             |   |   | otherwise
             |   |   |   | X_{i,2·j} ← Xc_{i-1,j-1}
             |   |   |   | X_{i,2·j+1} ← 1
             | X ← csort(X,0)
             | for j ∈ 1..4
             |   X_{n,2·j+1} ← 1
             | X_{n,1} ← 1
             | X
```

3. Empirical System Survivor Functions. (from MRL example worksheet)

4. Weibull Parameter Estimation Functions. (from MRL example worksheet)

5. "Semi-Parametric" System MRL Functions

   a) Numerator Constant $C(a,\beta,\alpha) := \int_{a}^{inf} S(t,\beta,\alpha)\,dt$

   b) KME MRL Function: (from MRL example worksheet)

   c) Smoothed KME MRL Function: (from MRL example worksheet)

   d) PEXE MRL Function: (from MRL example worksheet)

6. Function to determine the Empirical Reliability Goal given T

   determine the MRL function estimation technique: $switch := 1$

   swithch = 1: KME
   swithch = 2: PEXE
   swithch = 3: sKME

$$R_n(G,T,\mu,C) := \left\| \begin{array}{l} out \leftarrow S_{KME}\left(G,\mu - m_{KME}(G,T,C)\right) \ \text{if } switch = 1 \\ out \leftarrow S_{PEXE}\left(G,\mu - m_{PEXE}(G,T,C)\right) \ \text{if } switch = 2 \\ out \leftarrow S_{sKME}\left(G,\mu - m_{sKME}(G,T,\mu,C)\right) \ \text{if } switch = 3 \\ out \end{array} \right.$$

7. Reliability vs T

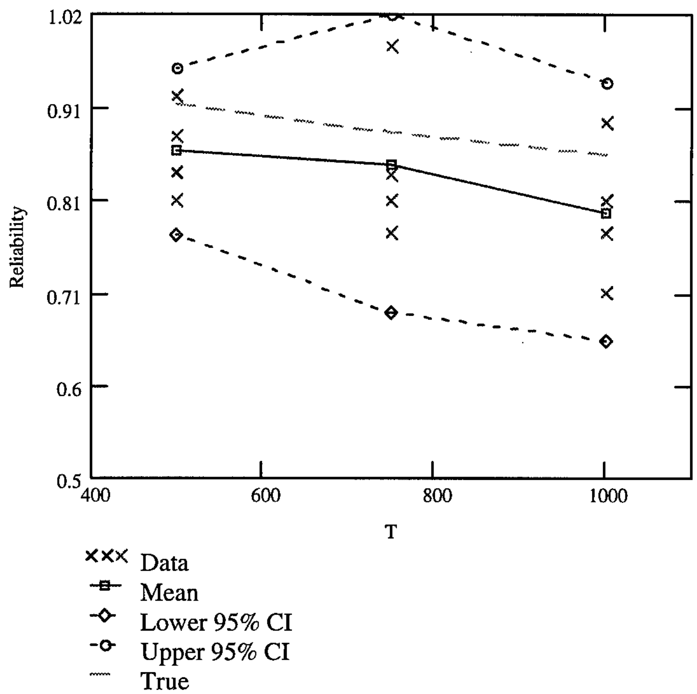
$$
\begin{aligned}
Rel_n := \;& \text{for } k \in 0..nreps-1 \\
& \quad Z \leftarrow DATA(\alpha,\beta) \\
& \quad b \leftarrow \beta_n(Z) \\
& \quad C \leftarrow C\left(Z_{n,0}, b, \alpha_n(Z,b)\right) \\
& \quad \mu \leftarrow m_{KME}(Z,0,C) \text{ if } switch = 1 \\
& \quad \mu \leftarrow m_{PEXE}(Z,0,C) \text{ if } switch = 2 \\
& \quad \mu \leftarrow \mu_{sKME}(Z) + C \text{ if } switch = 3 \\
& \quad i \leftarrow 0 \\
& \quad T \leftarrow Tmax \\
& \quad \text{while } T \geq Tmin \\
& \quad\quad period_{k,i} \leftarrow T \\
& \quad\quad rel_{k,i} \leftarrow R_n(Z,T,\mu,C) \\
& \quad\quad T \leftarrow T - Tstep \\
& \quad\quad i \leftarrow i + 1 \\
& npts \leftarrow i - 1 \\
& \text{for } j \in 0..npts \\
& \quad \text{for } i \in 0..nreps-1 \\
& \quad\quad out_{j\cdot(nreps)+i,0} \leftarrow period_{i,j} \\
& \quad\quad out_{j\cdot(nreps)+i,1} \leftarrow rel_{i,j} \\
& out
\end{aligned}
$$

$$
\begin{aligned}
avg := \;& imax \leftarrow \frac{Tmax - Tmin}{Tstep} \\
& \text{for } i \in 0,1..imax \\
& \quad jmin \leftarrow nreps \cdot i \\
& \quad jmax \leftarrow jmin + nreps - 1 \\
& \quad out_{i,0} \leftarrow mean\left(submatrix\left(Rel_n, jmin, jmax, 0, 0\right)\right) \\
& \quad out_{i,1} \leftarrow mean\left(submatrix\left(Rel_n, jmin, jmax, 1, 1\right)\right) \\
& \quad half \leftarrow \frac{qt(0.975, nreps-1)}{nreps^{\frac{1}{2}}} \cdot \frac{nreps}{nreps-1} \\
& \quad out_{i,2} \leftarrow out_{i,1} - half \cdot stdev\left(submatrix\left(Rel_n, jmin, jmax, 1, 1\right)\right) \\
& \quad out_{i,3} \leftarrow out_{i,1} + half \cdot stdev\left(submatrix\left(Rel_n, jmin, jmax, 1, 1\right)\right) \\
& out
\end{aligned}
$$

Theoretical reliability at each value of T in the empirical model

$$\text{Rel} := \left| \begin{array}{l} \text{for } i \in 0..\text{rows}(\text{avg}) - 1 \\ \quad \text{out}_i \leftarrow S_0\left(T_v\left(\text{avg}_{i,0}\right)\right) \\ \text{out} \end{array} \right.$$

$i := 0..\text{rows}\left(\text{Rel}_n\right) - 1$ $\quad\quad$ $j := 0..\text{rows}(\text{avg}) - 1$



Empirical Policy Costs

1. Model Inputs.

sample size:	$n \equiv 300$
exponential censor rate:	$\lambda \equiv .0018$
number of replications:	$\text{nreps} \equiv 4$
Weibull param. estimation:	
shape param initial guess:	$\beta_0 \equiv 1.3$
shape param tolerance:	$\varepsilon \equiv .02$
replacement period:	$\text{Tmax} := 2000$ $\quad$ $\text{Tstep} := 500$ $\quad$ $\text{Tmin} := 500$

2. Normalized <u>Non-parametric</u> ARP Cost Model

$$
C_{An}(G,T) := \left|
\begin{array}{l}
int \leftarrow T \ \text{ if } \ 0 \le T \le G_{1,0} \\
T \leftarrow G_{n,0} \ \text{ if } \ T > G_{n,0} \\
\text{if } \ T > G_{1,0} \\
\quad \left|
\begin{array}{l}
int \leftarrow G_{1,0} \\
i \leftarrow 2 \\
S \leftarrow 1 \\
a \leftarrow int \\
\text{while } \ a < T \\
\quad \left|
\begin{array}{l}
S \leftarrow S_{KME}\left(G, G_{i,0}\right) \ \text{ if } \ G_{i-1,1} = 1 \\
int \leftarrow int + S \cdot \left(G_{i,0} - G_{i-1,0}\right) \\
i \leftarrow i + 1 \\
a \leftarrow G_{i,0}
\end{array}
\right. \\
S \leftarrow S_{KME}\left(G, G_{i,0}\right) \ \text{ if } \ G_{i-1,1} = 1 \\
int \leftarrow int + S \cdot \left(G_{i,0} - T\right)
\end{array}
\right. \\
out \leftarrow \dfrac{1}{int} \\
out
\end{array}
\right.
$$

3. Normalized <u>Parametric</u> BRP Cost Model

$$
C_{Bn}\left(T, \beta_n, \alpha_n\right) := \frac{1 + \displaystyle\sum_{i=1}^{4} \frac{c_{f_i}}{c_p} \cdot W\left(T, \beta_{n_i}, \alpha_{n_i}\right)}{T}
$$

4. Normalized <u>Parametric</u> ORP Cost Model

$$
C_{On}(T, ageT, \beta, \alpha) := \left|
\begin{array}{l}
\tau \leftarrow \dfrac{T}{2} \\
\tau_{step} \leftarrow 100 \\
out_0 \leftarrow C_A(T) \\
out_1 \leftarrow \tau \\
\text{while } \tau \le T \\
\quad \left|
\begin{array}{l}
denom \leftarrow \tau + \displaystyle\int_0^{T-\tau} \exp\left[\left[\sum_{i=1}^{4} -\left[(\tau + u)\cdot\alpha_i\right]^{\beta_i} + \left(\tau\cdot\alpha_i\right)^{\beta_i}\right]\right] du \\
new \leftarrow \dfrac{1}{denom}\cdot\left(c_p + \displaystyle\sum_{i=1}^{4} c_{f_i}\cdot W\left(\tau, \beta_i, \alpha_i\right)\right) \\
\text{if } new < out_0 \\
\quad \left|
\begin{array}{l}
out_0 \leftarrow new \\
out_1 \leftarrow \tau
\end{array}
\right. \\
\tau \leftarrow \tau + \tau_{step}
\end{array}
\right. \\
out
\end{array}
\right.
$$

5. Empirical Policy Costs vs Reliability

```
COST_n := | for k ∈ 0.. nreps − 1
          |   | Z ← DATA(α, β)
          |   | i ← 0
          |   | T ← Tmax
          |   | for j ∈ 1.. 4
          |   |   | Z_c ← submatrix(Z, 0, n, 2·j, 2·j + 1)
          |   |   | β_cn_j ← β_n(Z_c)
          |   |   | α_cn_j ← α_n(Z_c, β_cn_j)
          |   | while T ≥ Tmin
          |   |   | period_k,i ← T
          |   |   | age_k,i ← C_An(Z, T)
          |   |   | block_k,i ← C_Bn(T, β_cn, α_cn)
          |   |   | temp ← C_On(T, age_k,i, β_cn, α_cn)
          |   |   | opp_k,i ← temp_0
          |   |   | tau_k,i ← temp_1
          |   |   | T ← T − Tstep
          |   |   | i ← i + 1
          | npts ← i − 1
          | for j ∈ 0.. npts
          |   for i ∈ 0.. nreps − 1
          |     | out_j·(nreps)+i,0 ← period_i,j
          |     | out_j·(nreps)+i,1 ← age_i,j
          |     | out_j·(nreps)+i,2 ← block_i,j
          |     | out_j·(nreps)+i,3 ← opp_i,j
          |     | out_j·(nreps)+i,4 ← tau_i,j
          | out
```

$$avg := \left| \begin{array}{l} imax \leftarrow \dfrac{Tmax - Tmin}{Tstep} \\ \text{for } i \in 0,1..imax \\ \quad \left| \begin{array}{l} jmin \leftarrow nreps \cdot i \\ jmax \leftarrow jmin + nreps - 1 \\ out_{i,0} \leftarrow mean(submatrix(COST_n, jmin, jmax, 0, 0)) \\ out_{i,1} \leftarrow mean(submatrix(COST_n, jmin, jmax, 1, 1)) \\ out_{i,2} \leftarrow mean(submatrix(COST_n, jmin, jmax, 2, 2)) \\ out_{i,3} \leftarrow mean(submatrix(COST_n, jmin, jmax, 3, 3)) \\ out_{i,4} \leftarrow mean(submatrix(COST_n, jmin, jmax, 4, 4)) \end{array} \right. \\ out \end{array} \right.$$

Theoretical ARP cost at each value of T in the empirical model

$$Age := \left| \begin{array}{l} \text{for } i \in 0..rows(avg) - 1 \\ \quad out_i \leftarrow C_A(avg_{i,0}) \\ out \end{array} \right.$$

Theoretical BRP cost at each value of T in the empirical model

$$Block := \left| \begin{array}{l} \text{for } i \in 0..rows(avg) - 1 \\ \quad out_i \leftarrow C_B(avg_{i,0}) \\ out \end{array} \right.$$

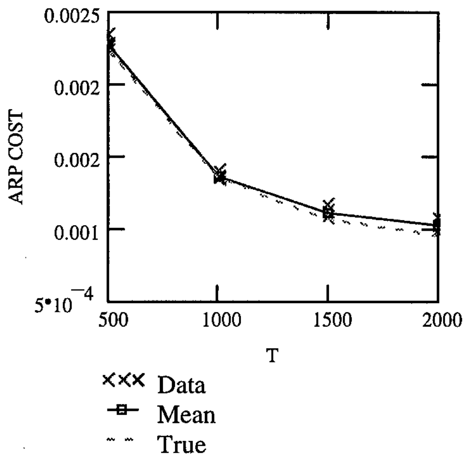
Theoretical ORP cost at each value of T in the empirical model

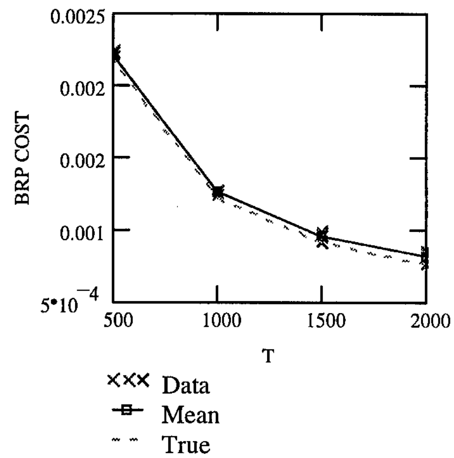
$$Opp := \left| \begin{array}{l} \text{for } i \in 0..rows(avg) - 1 \\ \quad out_i \leftarrow C_O(avg_{i,0})_0 \\ out \end{array} \right.$$

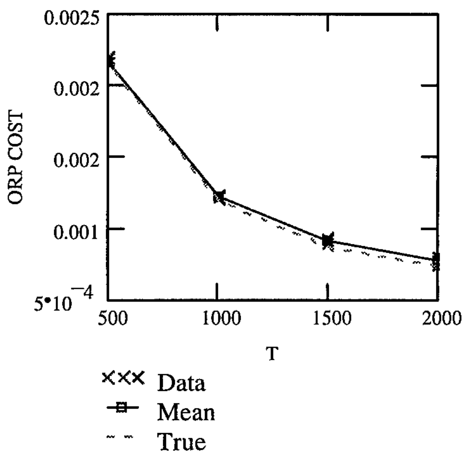
Theoretical ORP value of τ at each value of T in the empirical model

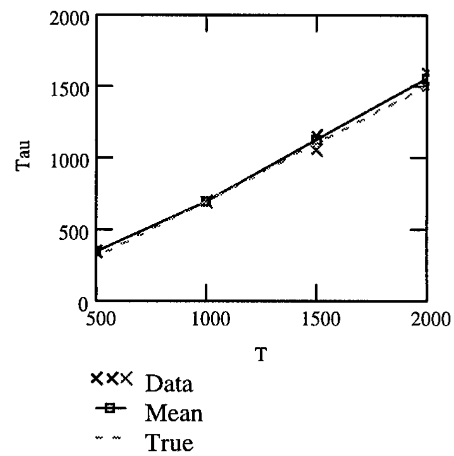
$$Tau := \left| \begin{array}{l} \text{for } i \in 0..rows(avg) - 1 \\ \quad out_i \leftarrow C_O(avg_{i,0})_1 \\ out \end{array} \right.$$

$i := 0..\,\text{rows}\left(COST_n\right) - 1$ $\qquad$ $j := 0..\,\text{rows}(avg) - 1$









Appendix D: AMRAAM Data

Z_i	δ_i	Z_i	δ_i	Z_i	δ_i	Z_i	δ_i	Z_i	δ_i	Z_i	δ_i	Z_i	δ_i	Z_i	δ_i	Z_i	δ_i
1	1	14.1	1	42.8	1	84.5	0	178.3	0	334.7	0	555.1	1	840.2	0	1611.6	0
1.2	0	14.2	0	43.1	0	85.5	0	178.7	1	334.7	0	555.8	0	844.4	0	1621.4	1
1.4	0	14.6	1	43.2	0	86.1	0	179.9	0	336.1	0	563.9	0	844.5	0	1750.5	0
1.4	0	15.2	0	43.2	0	86.4	0	180	0	346.7	1	564	0	845.8	0	1836	0
1.5	0	15.2	0	43.3	0	87.5	0	180.7	1	346.7	1	568.3	0	846.7	1	1854.3	0
1.6	1	15.4	0	43.3	0	87.9	1	181.5	1	348.8	1	571.2	0	849.8	0	1858.2	0
1.8	0	16.5	0	43.5	0	88	1	183.1	0	353.2	0	571.6	0	854	1	1861.8	0
1.9	0	16.9	0	43.5	0	88.3	0	190.9	1	355.7	0	574	1	861.9	0	1882.6	0
2.1	0	17	0	44	0	88.7	0	191.1	1	355.9	0	575.5	0	869	0	1907.8	0
2.1	0	17.1	0	44.4	1	90	0	193.8	0	357.8	1	575.6	1	872	0	1950	0
2.4	1	17.6	0	44.4	0	90.1	0	194.2	0	358.9	0	578.2	1	876.1	1	2011.3	0
2.4	1	17.9	0	44.5	0	92	1	194.3	0	360.5	1	578.2	0	877.5	0	2012.4	0
2.4	1	18	0	44.6	0	92.6	0	194.4	0	361.4	1	578.3	1	878.8	0	2024.8	0
2.4	0	18.6	1	45.1	0	93.4	0	194.4	0	364.4	0	578.9	1	883.9	0	2054.2	1
2.4	0	18.7	0	45.1	0	94.1	1	195.4	0	366	1	582.1	0	883.9	0	2070.1	0
2.7	0	19.4	0	45.6	1	95	0	195.6	0	367.1	1	582.2	1	884.3	0		
2.8	1	19.5	1	45.8	0	95.4	1	196.2	1	368.8	1	582.7	1	886.3	0		
2.8	0	19.5	0	46.3	0	96.2	1	199.3	0	372.6	0	585	0	888.9	0		
2.9	0	19.5	0	46.4	0	96.4	0	199.6	0	374.7	1	585.9	0	893	0		
3	1	19.6	0	46.9	0	96.5	1	201.5	0	375.9	0	591.9	0	893.8	0		
3	0	19.7	0	47.3	0	97.8	1	203.4	0	377.1	0	591.9	0	894.4	0		
3.1	0	19.8	1	47.3	0	99.2	1	204.5	1	377.3	1	592.7	0	899.9	1		
3.4	1	20	0	47.4	1	99.7	0	204.6	1	377.6	0	598.2	0	905.6	1		
3.5	0	20.1	0	47.8	0	100.1	0	207	1	377.9	0	598.7	0	923.6	0		
3.6	0	20.9	1	48.2	0	101	1	207	0	379.8	0	600.9	0	928.3	0		
3.7	0	21	1	48.4	0	101.5	0	208.7	0	380.2	0	603.6	0	931.1	1		
4	0	21	0	48.6	0	102	0	210.8	0	381.9	0	603.7	0	934	0		
4.1	0	22.1	0	48.7	0	102.2	0	211.8	0	382.1	1	612.1	0	939.7	0		
4.3	0	22.4	0	48.9	0	102.4	1	212.5	0	384	1	614.8	1	952.8	0		
4.5	0	22.5	1	49	0	103.5	1	212.6	0	384.3	1	614.8	0	982.2	0		
4.5	0	22.5	0	49.4	0	104.4	1	215.5	1	386.4	0	615.5	0	986.5	1		
4.5	0	22.6	0	49.6	1	104.4	1	216.5	1	386.9	0	616.9	0	995.2	1		
4.5	0	22.7	0	50.1	0	104.6	0	217.7	0	387.3	0	621.2	1	999	0		
4.7	0	22.8	0	50.5	0	104.8	0	218.9	0	387.5	0	624.1	0	1002.8	0		
4.7	0	23.2	0	50.9	0	106.4	0	219.1	0	390.2	0	624.5	0	1005.4	0		
4.8	0	23.4	0	50.9	0	107.4	0	219.6	1	392.9	0	628	0	1007.6	0		
5	0	23.5	1	52	0	107.7	0	220.1	0	402.1	1	630	0	1008.6	0		
5	0	23.9	0	52.3	1	108.8	0	221.2	1	404.9	1	640	0	1017.3	0		
5.1	0	24	0	52.4	1	109.2	0	221.4	0	405.5	1	650.7	1	1018.5	0		
5.2	1	24.1	0	53	0	109.6	0	221.7	1	407.4	0	654.5	0	1022.4	1		
5.3	1	24.1	0	53.1	0	109.8	0	224.6	0	408.7	1	656.7	0	1025.9	0		
5.5	0	24.2	0	53.8	0	111	0	224.9	0	409.8	1	659.3	0	1030.5	0		
6	1	24.2	0	54.4	0	111.2	0	225.1	0	416.6	1	660	1	1032.1	0		
6	0	24.2	0	54.6	0	111.5	1	225.4	0	419.5	1	664.8	1	1034.9	1		
6	0	24.2	0	55	0	111.8	1	227.9	0	426.4	0	665.4	0	1039.4	0		
6.2	0	25.3	1	55.2	1	111.8	0	232.2	0	426.6	1	666.6	1	1042.6	0		
6.4	0	25.6	1	56	1	112.3	1	232.5	0	426.9	1	667.9	1	1044.3	1		
6.4	0	25.6	0	56.9	1	113.3	0	235.6	0	428.9	1	671.1	0	1053.9	0		
6.4	0	25.9	0	57.1	0	114.2	0	236.9	0	430.3	1	676.9	1	1054.7	0		
6.5	0	27.1	0	57.7	0	114.3	0	238.1	0	436.3	0	684.4	1	1057	1		
6.5	0	27.6	0	57.9	1	115	0	238.2	0	444	1	685.5	0	1066.7	0		
6.5	0	28	0	58.1	0	115.8	0	238.5	1	444.2	1	690.9	1	1071.3	0		
7	1	28	0	58.8	0	116.1	0	243.6	0	445.7	0	694	0	1082.7	0		
7	0	28.6	0	59.4	0	116.7	0	244.9	0	446	0	698	1	1089.5	0		
7.3	0	29	0	60.4	0	119.6	0	246.5	1	452.8	0	702.7	0	1091.6	0		
7.5	0	29.2	0	61	1	120.7	1	247.5	0	453.3	0	708.1	0	1092.9	0		
7.6	1	30	0	61.9	0	121.8	0	248.8	0	455.1	1	714.4	1	1096.6	0		

7.6	0	30.5	0	62.4	0	123	1	249.7	0	456	1	715.3	0	1098.1	0
7.6	0	30.5	0	63.4	0	123.2	0	250.1	0	456.6	1	716.8	0	1114.9	0
7.6	0	30.5	0	63.8	0	124.9	0	250.7	0	457.7	1	717.6	1	1115.6	1
7.6	0	30.6	0	63.8	0	128.2	0	253.8	0	463.4	0	718	0	1119.6	0
7.8	0	31.1	0	64.3	0	129.2	0	254.8	0	463.5	1	720.9	0	1124.8	1
8	0	31.2	0	64.4	1	129.3	1	256.4	0	464.7	1	721	0	1130.6	0
8	0	31.2	0	65.7	0	129.3	0	258.2	0	466	1	721.7	0	1134.1	0
8	0	31.2	0	66.5	1	131	1	260.4	1	467	0	734.2	0	1141.3	0
8.4	0	31.7	0	66.9	0	133.6	1	260.6	0	467.3	0	739.6	1	1142.8	1
8.6	1	32.3	0	67.3	0	135.9	0	262.3	1	468.1	0	742.7	0	1142.9	0
8.6	0	33.2	0	67.4	0	137.3	0	267.5	0	476.7	0	746.6	0	1153.3	0
8.6	0	33.3	0	67.6	0	138.3	0	270.4	1	477.2	0	748.3	0	1155.6	0
9	0	33.5	0	67.9	0	139	0	272.8	0	479.9	0	749.5	0	1161.6	0
9	0	33.7	0	68.4	0	139.7	0	272.8	0	481.2	0	751.9	0	1164.1	1
9.2	1	33.8	0	68.4	0	140.8	0	274.1	1	484.6	1	753.1	0	1170.2	0
9.3	0	33.9	0	68.5	1	143.7	1	275.6	1	490.2	0	755.3	0	1183.8	0
9.5	0	34	0	69.4	0	143.7	0	278.4	0	496.5	1	756.9	0	1191.5	0
9.5	0	34.3	0	69.9	0	144.6	0	279.7	0	497.2	1	760	0	1199.4	0
10.2	0	34.5	1	69.9	0	146.8	1	286	1	499.5	0	761.1	0	1219.6	0
10.5	0	34.5	0	70.1	0	147	1	288	0	500.1	0	765.9	0	1228.7	0
10.9	0	34.7	0	71	0	147.5	0	288.6	1	506.6	0	769.4	0	1248.4	1
11	0	34.9	0	71.5	0	147.6	0	289.9	1	511.8	1	772.3	1	1249	1
11.2	0	34.9	0	72.4	0	148.1	0	291	0	525.6	0	779.1	0	1252.4	0
11.2	0	35	0	73.4	0	149.7	0	293.2	0	525.7	0	779.6	0	1253.8	0
11.4	0	35.4	0	74	0	152.9	1	293.9	1	526.7	1	786	0	1257.8	0
11.4	0	35.5	0	74.2	0	153.4	0	294.1	0	529.5	0	792.6	0	1259.2	0
11.4	0	35.6	1	75.4	0	156.1	0	294.9	0	531	0	802.6	0	1259.3	0
11.9	1	35.6	0	75.6	0	157.8	1	300.4	0	531.6	0	802.8	1	1277.8	0
11.9	0	36	1	75.7	0	158.5	0	303.2	1	531.9	0	807.7	0	1305.2	0
11.9	0	36.6	0	75.9	0	158.9	0	304.3	0	532.2	0	808.5	0	1308	0
11.9	0	36.8	0	76.6	0	162.3	0	305.6	1	532.4	1	810.6	0	1327.7	1
12.8	0	37	0	77.1	0	162.7	1	306.2	1	534.9	0	813.6	0	1336.5	1
13	0	37.2	1	77.9	0	163.4	0	306.8	1	537.6	0	818.3	1	1340.4	1
13	0	38.7	0	80	0	164	0	309.6	1	537.8	1	821.5	0	1350.7	0
13.2	0	38.8	0	80	0	164.7	1	309.7	0	538.5	0	821.7	0	1358.2	1
13.4	0	39.7	1	80.2	0	165.4	0	315.4	0	538.8	0	823	0	1381.7	0
13.5	0	40	1	82.1	1	166.1	0	321.3	0	539.2	0	823.9	1	1393.9	0
13.5	0	40	0	82.9	1	169.6	0	321.5	0	541.2	0	829.1	0	1412.6	0
13.5	0	40.5	0	82.9	0	173.1	1	325	0	542.8	1	831.7	0	1461.1	0
13.5	0	40.6	0	83.3	0	173.7	0	328.3	0	544.3	0	834.8	1	1529.1	0
13.5	0	40.9	0	83.7	0	174.4	1	330.5	0	544.4	0	836.7	1	1556	0
13.9	0	42	0	83.8	0	175.6	0	332	0	551.5	1	837.2	1	1566.5	1
14	0	42.6	0	83.8	0	177.7	1	333.6	0	554.7	1	838.7	0	1581.1	0

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Vita

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13. ABSTRACT *(Maximum 200 words)*
The overall objective of this research effort was to formulate a preventive maintenance strategy for AMRAAM missiles subject to extended captive carry flight time. A preventive maintenance policy is only applicable if the item in question is aging, or deteriorating with time. Therefore, a supporting objective of this research is to characterize the aging process of the missile system through a non-parametric analysis of its Mean Residual Life (MRL) function. Three non-parametric, censored-data MRL function estimation techniques discussed in the literature are examined via a numerical example. All three estimation techniques provide MRL functions that exhibit greatly exaggerated decreasing trends compared to the MRL function of the underlying distribution in the example. A semi-parametric technique for estimating the MRL function is developed that shows dramatic improvement over the non-parametric results. Although the MRL analysis of the current AMRAAM failure data failed to provide evidence that the missile system is aging, three preventive maintenance policies discussed in the literature are investigated. The traditional approach of preventive maintenance policy optimization via cost function minimization requires the cost of a system failure be explicitly known. However, the penalty for a system failure is often subjective and difficult to express in monetary terms. A "reliability cost model" is developed whereby system reliability for each policy is expressed as a function of cost. This technique allows a direct assessment of the trade-off between cost and projected system reliability. Theoretical results are presented and the performance of the model applied to empirical data is assessed via a Monte Carlo simulation.

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